# AI4Math: How Machines Explore, Conjecture, and Discover Mathematics

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ZUSE INSTITUTE BERLIN

Berlin Mathematics Research Center MATH+

# What is this talk about?

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### Outline

- AI and Mathematics
- The Hadwiger-Nelson Problem
- Hilbert's 16th Problem

(Hyperlinked) References are not exhaustive; check references contained therein.

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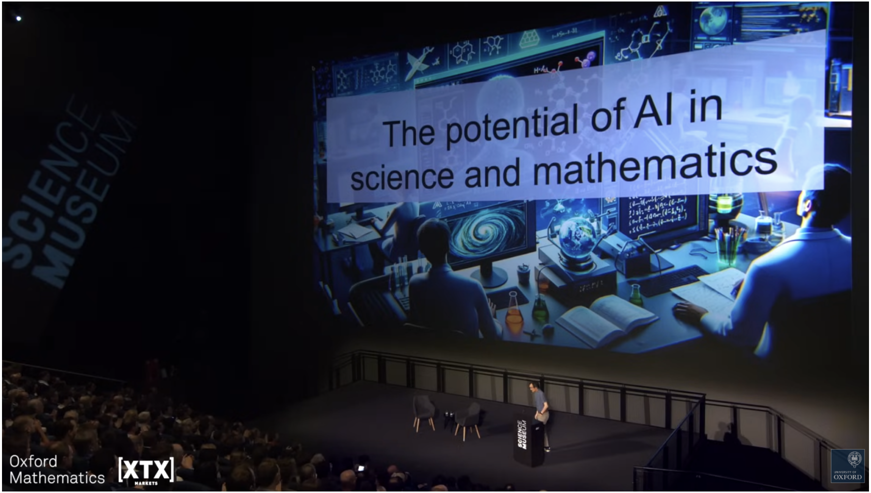
## AI and Mathematics




[The Potential for AI in Science and Mathematics - Terence Tao]

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**Various levels of co-creation.** [Haase and Pokutta, 2026]

- Digital Pen: basically like autocorrect, bibtex lookup, etc. “2000s”
- AI Task Specialist: ChatGPT, Claude, Gemini, etc. 2022 - 2025
- AI Assistant: Agents with integrated tools, verification, etc. 2025 - present
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Capabilities are impressive but unstable:

- SOTA models achieve post-PhD level scores on benchmarks, yet in day-to-day use make trivial (logical) errors.
- No hard verification of results and randomness across runs.
- Prompting and scaffolding are still a challenge.
- Availability of tools for verification etc crucial.

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## Mathematics with computers is not new

**Various high-profile examples from the past.**

- Four Color Theorem: massive computer-based case checking

  [Appel and Haken, 1977, Robertson et al., 1997]
- Kepler Conjecture / Hales' Theorem: extensive computer verification

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- Classification of Finite Simple Groups: Formal verification with Lean and Coq
- Boolean Pythagorean Triples Problem: A spectacular 200TB SAT-solver proof.

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**Crucial role in computational mathematics / scientific computing**

- Finite Elements
- Numerical Simulations
- Optimization
- Engineering
- ...

# The Hadwiger-Nelson Problem

joint work with: Aldo Kiem, Konrad Mundinger,
Christoph Spiegel, Max Zimmer

ICML 2025 (oral)
`https://arxiv.org/abs/2404.05509`

Partially supported by ExC MATH+ Project EF-LiOpt-3
Agent AI in Mathematics

[Mundinger et al., 2025]

# The Hadwiger-Nelson Problem

Problem (Nelson 1950, also: Gardner, Moser, Erdős, Harary, Tutte, ...)

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Infinite graph with vertex set $\mathbb{E}^2$ and edges $\{x, y\}$ for any $x, y \in \mathbb{E}^2$ with $\|x - y\| = 1$

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*Assuming* `Axiom of Choice (AoC)`*:* [Bruijn and Erdos, 1951]

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This problem has a long and complicated history... over 14 pages in [Soifer, 2024]

**Table 3.1** Who created the chromatic number of the plane problem?

| Publication | Year | Author(s) | Problem creator(s) or source named |
|---|---|---|---|
| [Gar2] | 1960 | Gardner | "**Leo Moser** ...writes..." |
| [Had4] | 1961 | Hadwiger (after Klee) | **Nelson** |
| [E61.22] | 1961 | Erdős | "I cannot trace the origin of this problem" |
| [Cro] | 1967 | Croft | "A long[18]-standing open problem of **Erdős**" |
| [Woo1] | 1973 | Woodall | **Gardner** |
| [Sim] | 1976 | Simmons | **Erdős**, **Harary**, and **Tutte** |
| [E80.38] [E81.23] [E81.26] | 1980–1981 | Erdős | **Hadwiger** and **Nelson** |
| [CFG] | 1991 | Croft, Falconer, and Guy | "Apparently due to **E. Nelson**" |
| [KW] | 1991 | Klee and Wagon | "Posed in 1960–61 by **M. Gardner** and **Hadwiger**" |

p. 24 in [Soifer, 2024]

# Lower bounds on $\chi(\mathbb{E}^2)$

# Lower bounds through unit distance graphs

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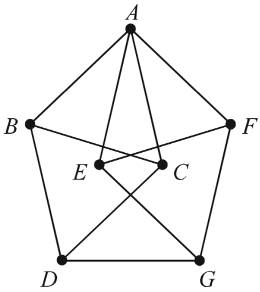

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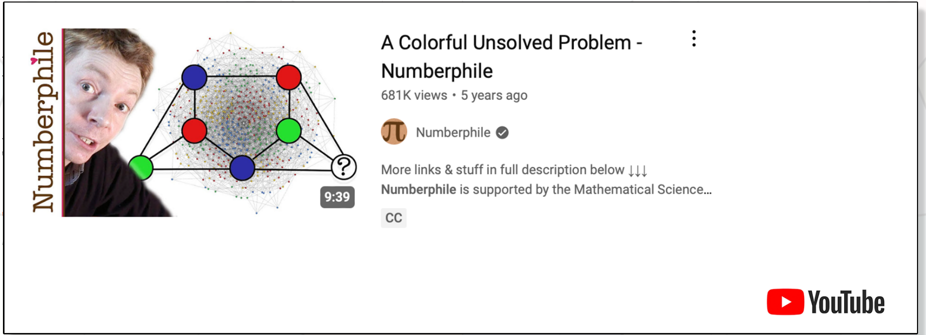

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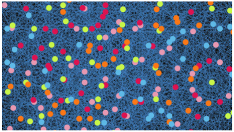

GRAPH THEORY

**Decades-Old Graph Problem Yields to Amateur Mathematician**

*By* EVELYN LAMB | APRIL 17, 2018 | 26 |

...number of vertices? The problem, now known as the Hadwiger-Nelson problem or the problem of finding the chromatic number of the plane, has piqued the interest of many mathematicians, including...

Quantamagazine

# Lower bounds through unit distance graphs

## The Hadwiger-Nelson Problem

Find unit distance graphs of large chromatic number.

Aubrey de Grey and Alexander Soifer, *Il Vicino*, January 18, 2020

Ronald L. Graham presents Aubrey D.N.J. de Grey the Prize: $1000, San Diego, September 22, 2018

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Simpler constructions with...

1. 1581 vertices — for detail see [De Grey, 2018]
2. 627 vertices — [Exoo and Ismailescu, 2020]
3. 553 vertices (as part of `Polymath16`) — Marijn Heule, for details see [Mixon, 2021]
4. 509 vertices (as part of `Polymath16`) — Jaan Parts, for details see [Mixon, 2021]

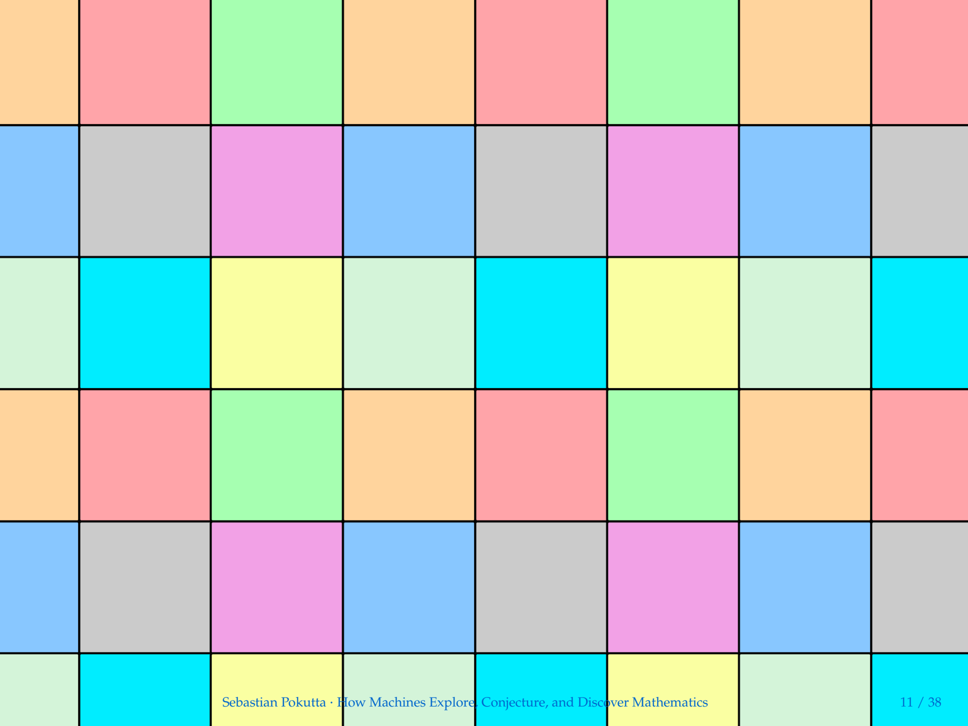
# Upper bounds on $\chi(\mathbb{E}^2)$

# Upper bounds through colorings

## The Hadwiger-Nelson Problem

Explicit colorings $g : \mathbb{E}^2 \to [c] := \{1, \ldots, c\}$, usually derived through tesselations using simple polytopal shapes, give

$$5 \leq \chi(\mathbb{E}^2) \leq \ldots$$



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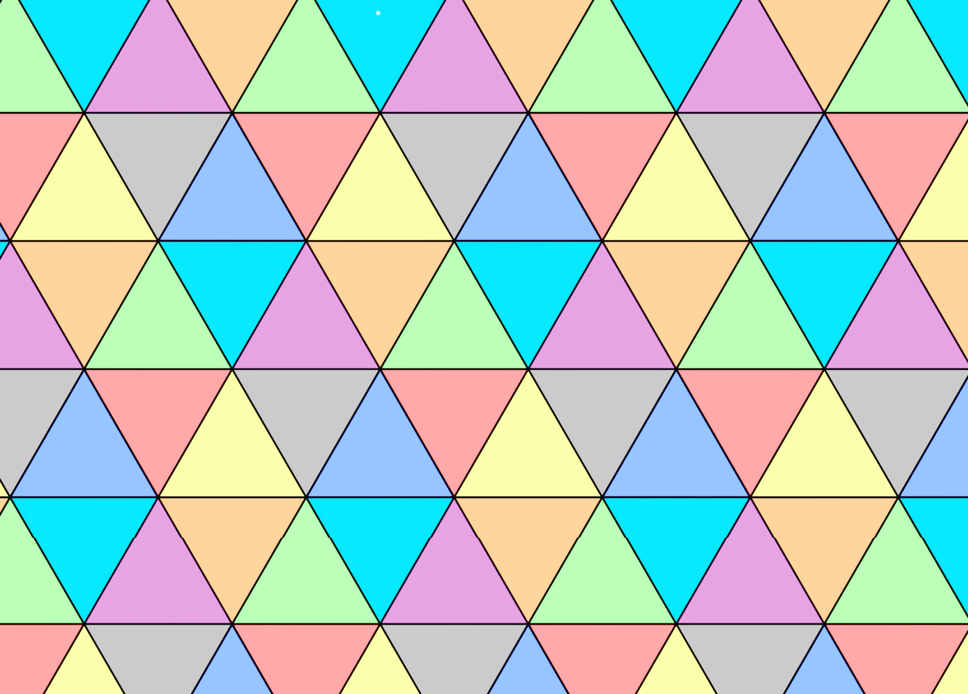
$$5 \leq \chi(\mathbb{E}^2) \leq 9.$$



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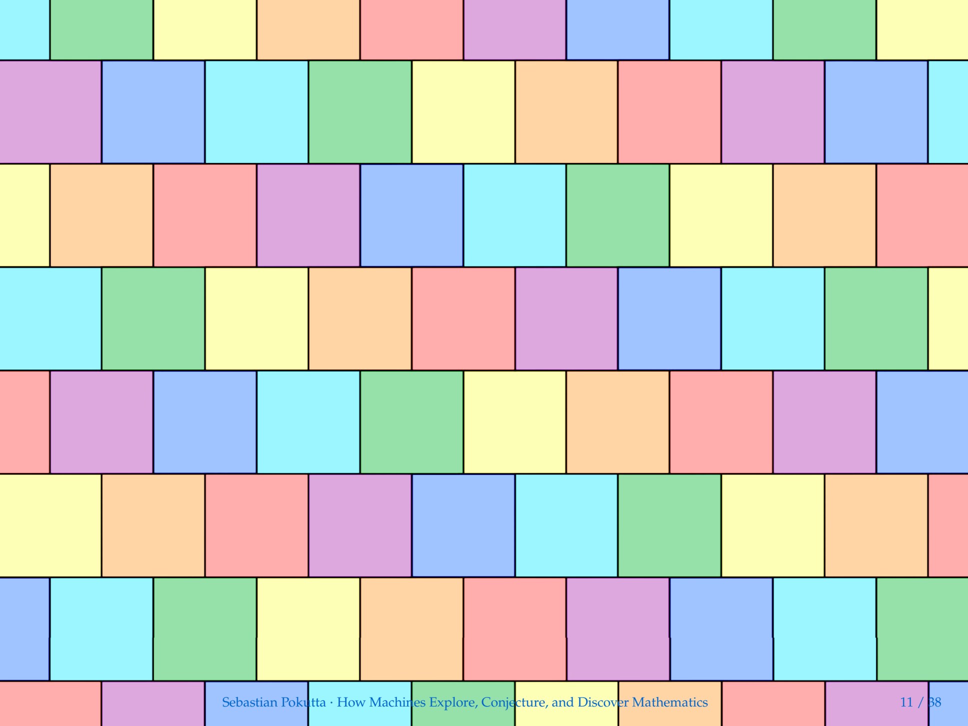

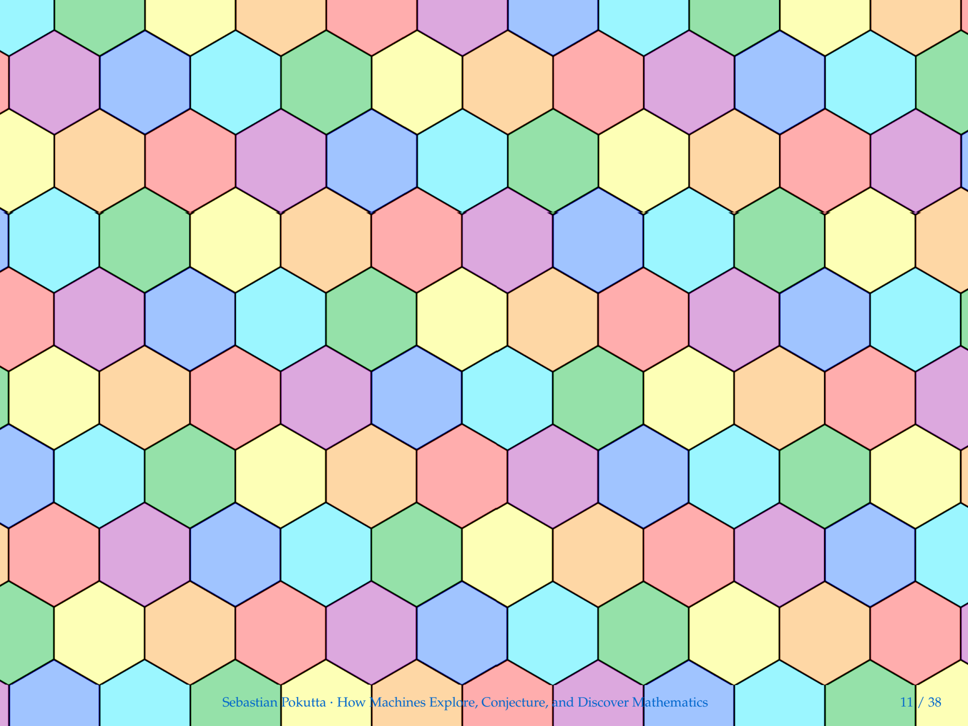

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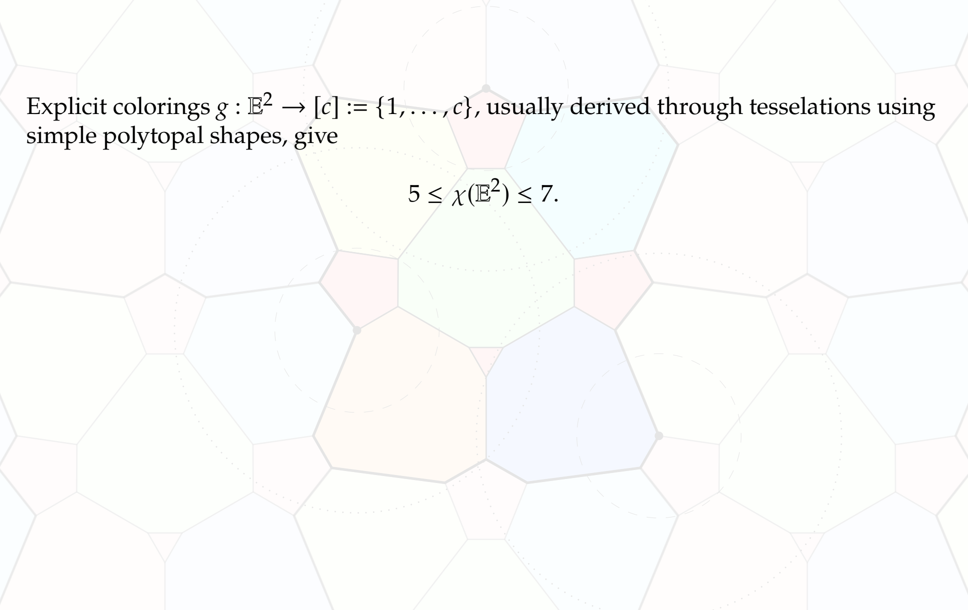

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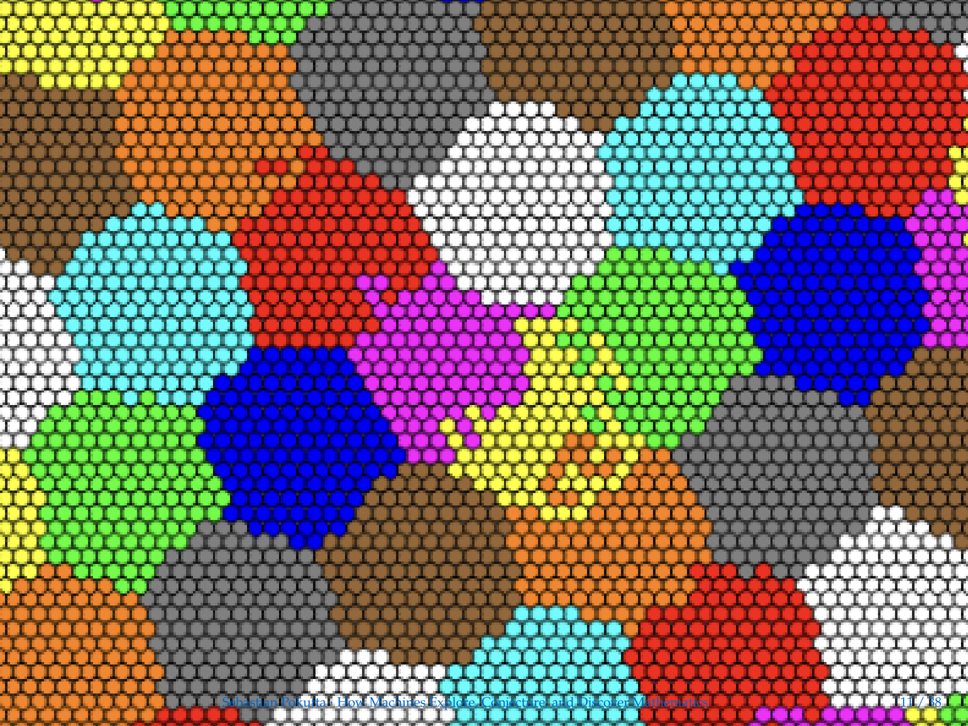
**Question.** Can we use computers to find admissible colorings $g : \mathbb{E}^2 \to [c]$, i.e.,

$$\left\{x \in \mathbb{E}^2 \mid \exists y \in B_1(x) : g(x) = g(y)\right\} = \emptyset?$$

... attempts, e.g., via discretization and SAT solvers...



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**Idea.** Use a parameterized and easily differentiable family $g_\theta : \mathbb{E}^2 \to \Delta_c$ and find

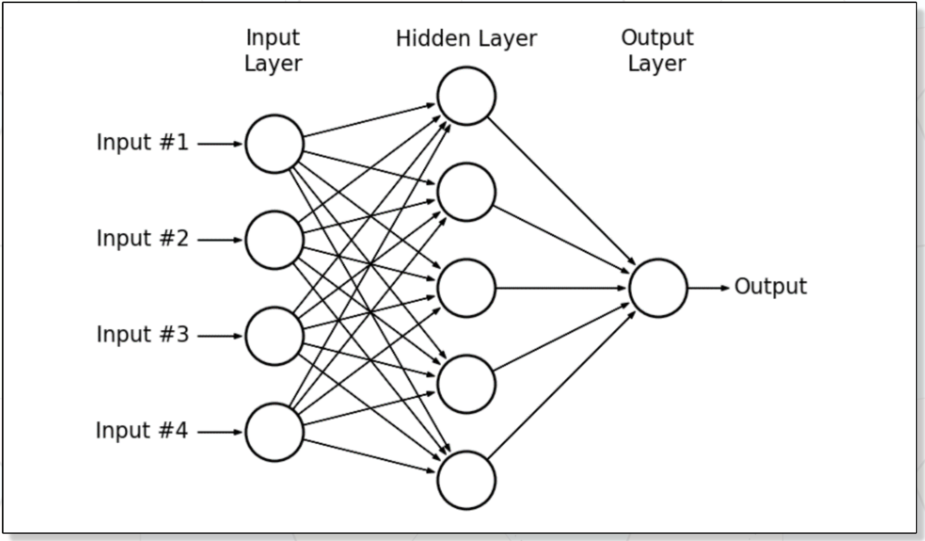
$$\arg\min_\theta \mathbb{E}\left[\int_{B_1(x)} g_\theta(x) \cdot g_\theta(y)\, dy \,\middle|\, x \in \mathbb{E}^2\right].$$

**Key Point.** Approach is continuous in nature.

# New upper bounds via machine learning?

# One second recap: Neural Networks

## Neural Networks as Colorings



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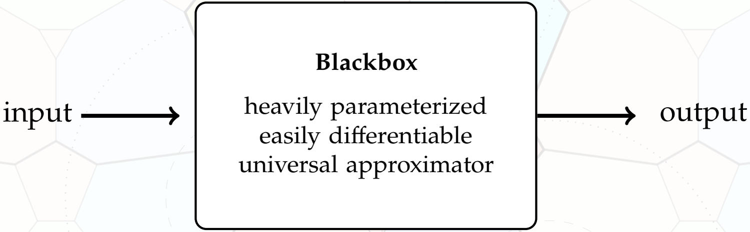
**Here.** Simply a parameterized continuous function to model the coloring.



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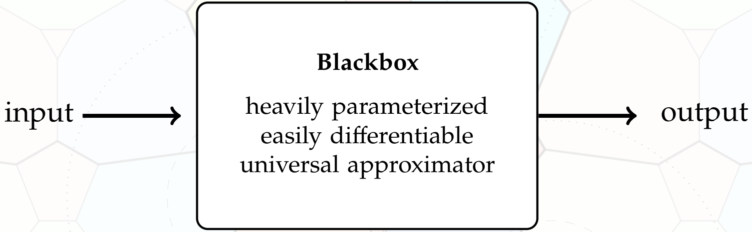


**Theorem (Universal Approximation Theorem)**

*Feedforward neural networks with certain activation functions are dense (w.r.t. compact convergence) in the space of continuous functions.*

# Can we improve the upper bound?

## Neural Networks as Colorings

**Idea.** Use gradient descent to train a feedforward network $g_\theta$ to minimize

$$L(\theta) = \int_{[-b,b]\times[-b,b]} \int_{B_1(x)} g_\theta(x) \cdot g_\theta(y)\, dy\, dx$$

for some reasonable $b \in \mathbb{R}$?

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$$\nabla_\theta L(\theta) \approx \hat{\nabla}_\theta L(\theta) \doteq \frac{1}{m} \sum_{i=1}^{m} \nabla_\theta\, g_\theta(x^{(i)}) \cdot g_\theta(y^{(i)}),$$

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$$\theta_{k+1} = \theta_k - \alpha_k\, \hat{\nabla}_\theta L(\theta).$$

**⇒ Very flexible approach "Deep Annealing"**
(also: tropicalization of loss function aka softmax... "minimize the max")

# Unfortunately this coloring was already known...

## Neural Networks as Colorings

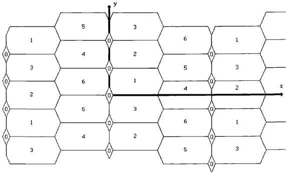
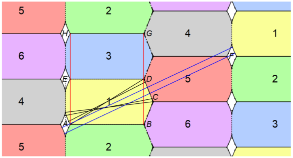
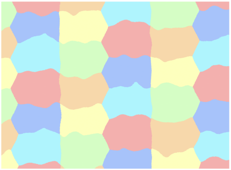


FIG. 3. A good 7-coloring of $(\mathbb{R}^2, 1)$.



### Theorem

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[Pritikin, 1998, Parts, 2020]

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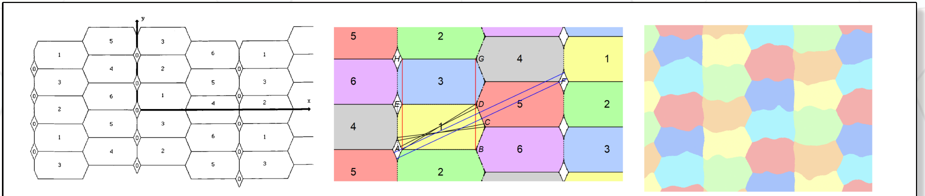

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**Corollary**

*Any unit distance graph with chromatic number 7 must have at least 6 992 vertices.*

⇒ **While coloring was known already maybe on the right track?**

# Off-diagonal variant

# Going off-diagonal

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Problem (Soifer in Nash and Rassias' *Open Problems in Mathematics*)

*Determine the continuum of six-colorings* $X_6 = \{d \mid (1, 1, 1, 1, 1, d)$ *can be realized*$\}$.

[Soifer, 1994a, Nash and Rassias, 2016]

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**Status.** Six-colorings exist for:

1. $d = 1/\sqrt{5}$ [Soifer, 1992]
2. $d = \sqrt{2} - 1$ [Hoffman and Soifer, 1993, 1996]
3. Part of family with $0.414 \approx \sqrt{2} - 1 \leq d \leq 1/\sqrt{5} \approx 0.447$ [Hoffman and Soifer, 1996, Soifer, 1994b, 2009]

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if color $i$ does not contain any points at distance $d_i$.

**Problem (Soifer in Nash and Rassias' *Open Problems in Mathematics*)**

*Determine the continuum of six-colorings* $X_6 = \{d \mid (1, 1, 1, 1, 1, d) \text{ can be realized}\}$.

[Soifer, 1994a, Nash and Rassias, 2016]

**Status.** Six-colorings exist for:

1. $d = 1/\sqrt{5}$ [Soifer, 1992]
2. $d = \sqrt{2} - 1$ [Hoffman and Soifer, 1993, 1996]
3. Part of family with $0.414 \approx \sqrt{2} - 1 \leq d \leq 1/\sqrt{5} \approx 0.447$ [Hoffman and Soifer, 1996, Soifer, 1994b, 2009]

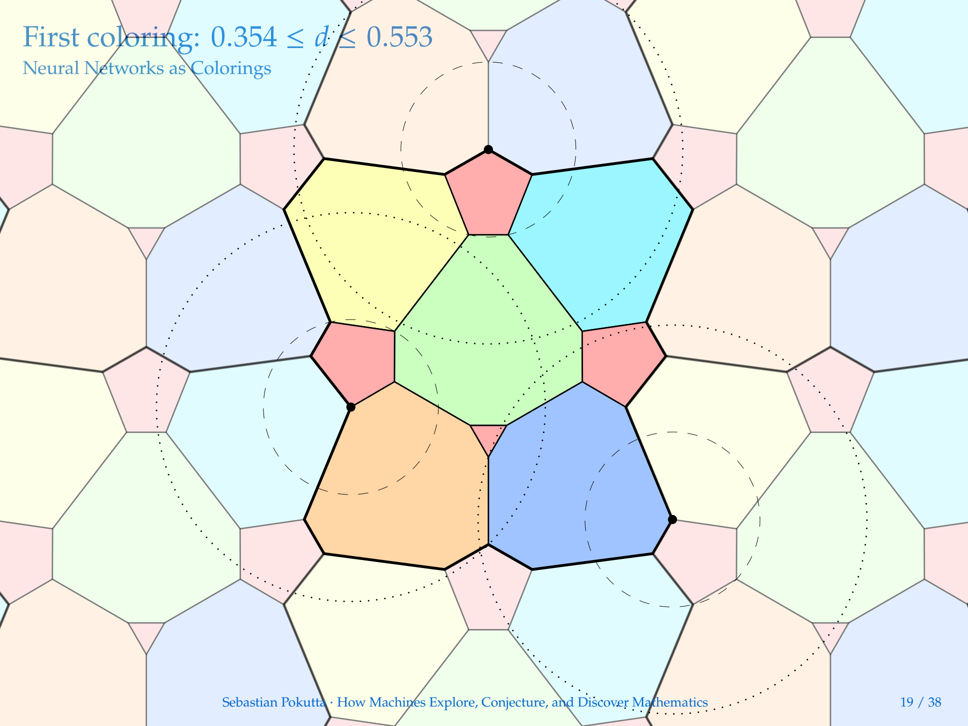
**Deep Annealing approach provides two new colorings leading to...**

**Theorem**

*$X_6$ contains the closed interval [0.354, 0.657].* [Mundinger et al., 2024, 2025]

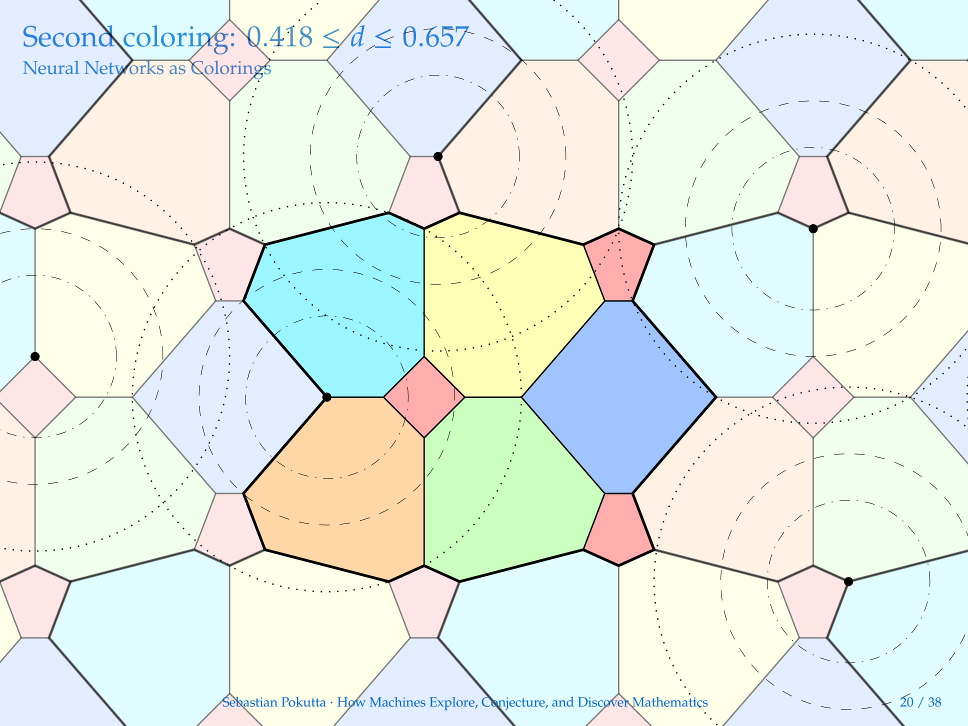
# First coloring: $0.354 \leq d \leq 0.553$

Neural Networks as Colorings



# Second coloring: $0.418 \leq d \leq 0.657$

Neural Networks as Colorings



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**Geombinatorics**

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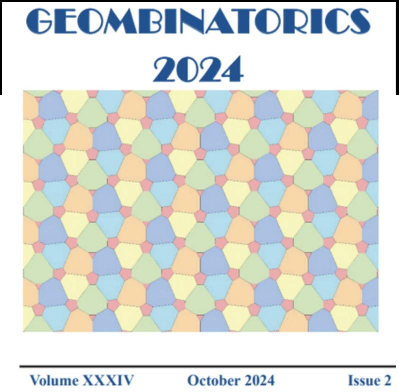
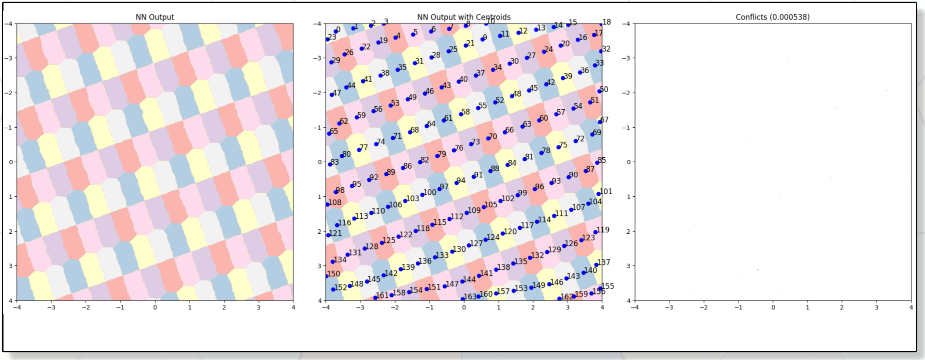


# Just numerics...?

## Neural Networks as Colorings



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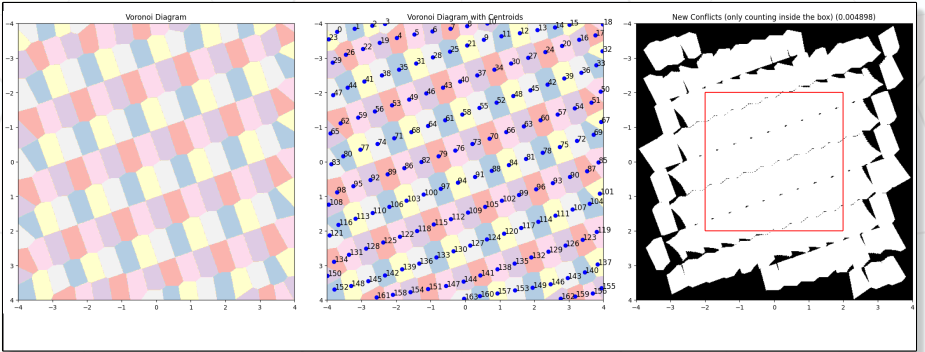

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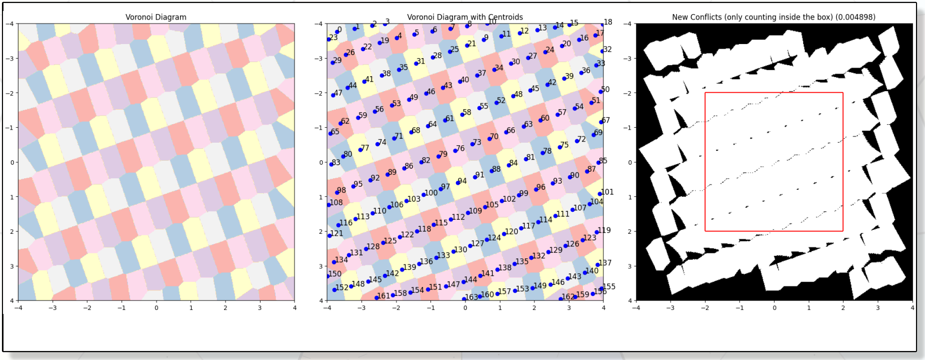
## Neural Networks as Colorings

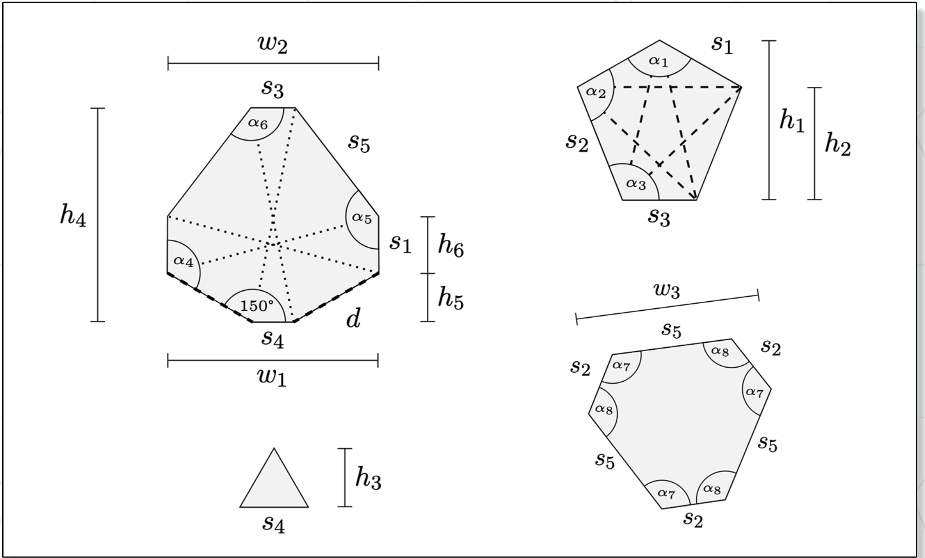



Voronoi cell filtering...

⇒ Exact constructions for both colorings.

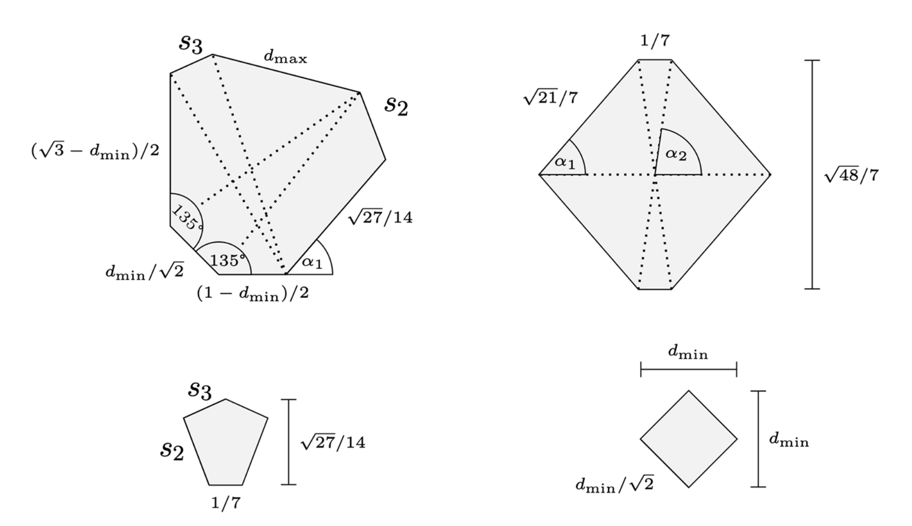
# First coloring: exact components

## Neural Networks as Colorings



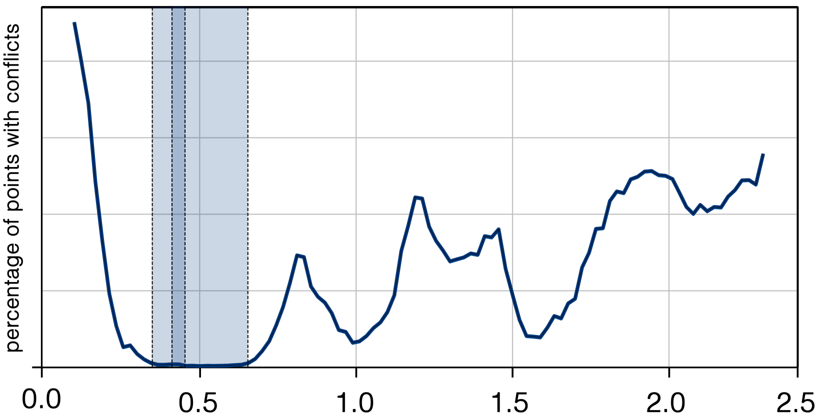
# Second coloring: exact components

## Neural Networks as Colorings



# Is this optimal?

## Neural Networks as Colorings




Numerical results showing the percentage of points with some conflict for a given forbidden distance $d$ in the sixth color found over several runs.

# 121 Patchworked Curves of Degree Seven

joint work with: Zoe Geiselmann, Michael Joswig, Lars Kastner,
Konrad Mundinger, Christoph Spiegel,
Marcel Wack, and Max Zimmer

Preprint (Sneak Peak)
`https://arxiv.org/abs/2602.06888`

Partially supported by ExC MATH+ Project EF-LiOpt-1
Neural Generative Models for Algebraic Curves

[Geiselmann et al., 2026]

In 1900, Hilbert formulated his famous 16th problem:

*Provide a topological classification of real plane algebraic curves of arbitrary degree d.*

[Hilbert, 1900]

In 1900, Hilbert formulated his famous 16th problem:

> *Provide a topological classification of real plane algebraic curves of arbitrary degree $d$.*

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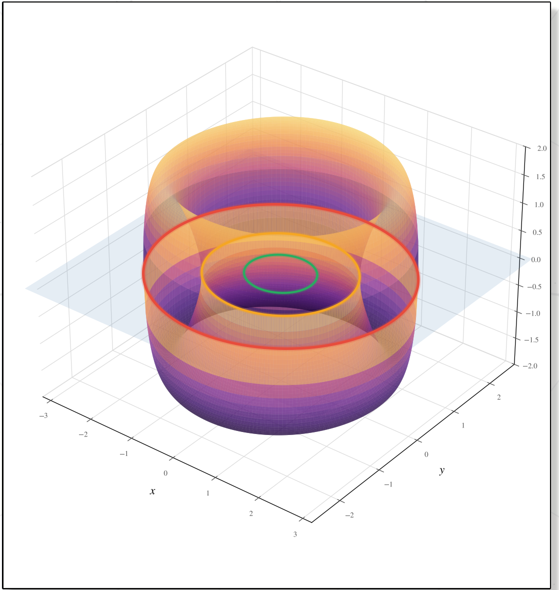
**Example.** For degree $d = 6$, the equation

$$z^2 = -\left[\left(\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - 1\right)\left(\frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} - 1\right)\left(\frac{x^2}{a_3^2} + \frac{y^2}{b_3^2} - 1\right)\right]$$

defines a real algebraic curve in the projective plane.

# Hilbert's 16th Problem

## Topological Characterization of Curves



# Current State-of-the-Art

## Topological Characterization of Curves

1. $d = 6, 7$: resolved in 1970s and 1980s [Rohlin, 1978, Nikulin, 1980, Viro, 1984, Gudkov, 1974]
2. $d = 8$: mostly but not entirely resolved [Orevkov, 2002]
3. $d = 9$: mostly fragmented [Orevkov, 2010]

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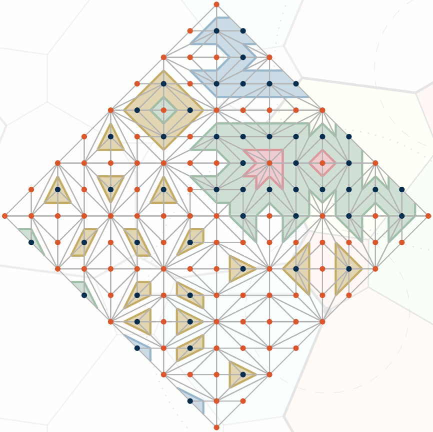
[Viro, 1986]

**Itenberg and Viro.** *"All real schemes of curves of degree $< 6$ and almost all real schemes of curves of degree 7 have been realized by the patchwork construction described above. On the other hand, there exist real schemes realizable by algebraic curves of some (high) degree, but not realizable by T-curves of the same degree. Probably such a scheme can be found even for degree 7 or 8."*

[Itenberg and Viro, 1996]

# Viro's Patchworking

## Examples in degree 8



(a) $\langle 17 \sqcup 1\langle 2\rangle \sqcup 1\langle 1\rangle\rangle$ (b) $\langle 16 \sqcup 3\langle 1\rangle\rangle$

**Note.** Curve is defined by a (regular) triangulation and a signing with ±1 at the nodes; actual polynomial obtained via lifting.

[Geiselmann et al., 2026]

## Theorem (Informal)

*All real schemes of curves of degree 7 can be realized by patchwork constructions.*

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*All real schemes of curves of degree 7 can be realized by patchwork constructions. Moreover, 4 triangulations are sufficient to cover all types.*

# Answering Itenberg and Viro's Open Problem for Degree 7

## Our results

[Geiselmann et al., 2026]

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[Geiselmann et al., 2026]

### Theorem (Informal)

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### Theorem (Informal)

*All real schemes of curves of degree 6 can be realized by patchwork constructions with no more than two 2 triangulations.*

Moreover,

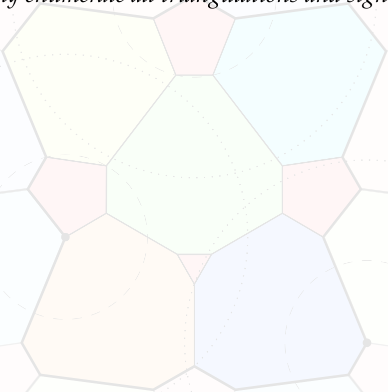
1. constructions are explicit
2. the lifting coefficients are very small
3. we provide explicit distributions over the realized types by triangulations
4. ... (see paper for more properties)

# What does this have to do with AI?

## Our results

In principle the problem is “trivial”:

*Simply enumerate all triangulations and signings.*



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“Minor details”:

1. For a single triangulation there are $2^{36} = 68.719.476.736$ signings. At initially 20 type computations/sec $\approx$ 100 cpu-days for a single triangulation.
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2. Then still need to "find" good triangulations.

**$\Rightarrow$ Basically impossible to search**

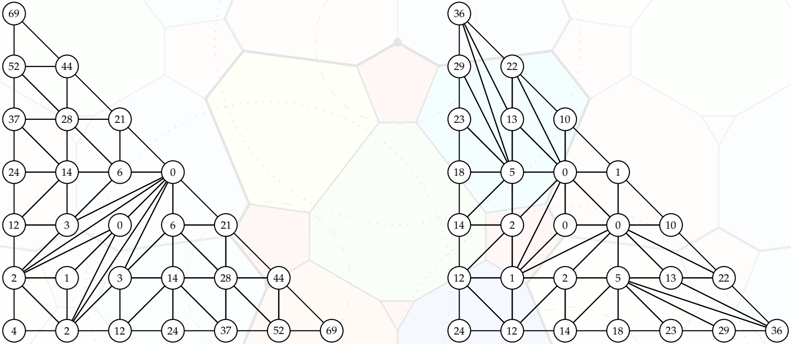
Our approach: combination of human + AI insights

1. New highly-optimized type algorithm developed leveraging AI, symmetry reduction, etc with final implementation in "near-assembler" Rust
$\Rightarrow \approx 4 \cdot 10^6$ type computations/sec or $\approx$ 4 hours for a single triangulation
2. (Many) intelligent guesses by humans for good triangulations

(more details soon...)

# Two Regular Triangulations to rule them all

## Characterization of curves of degree six

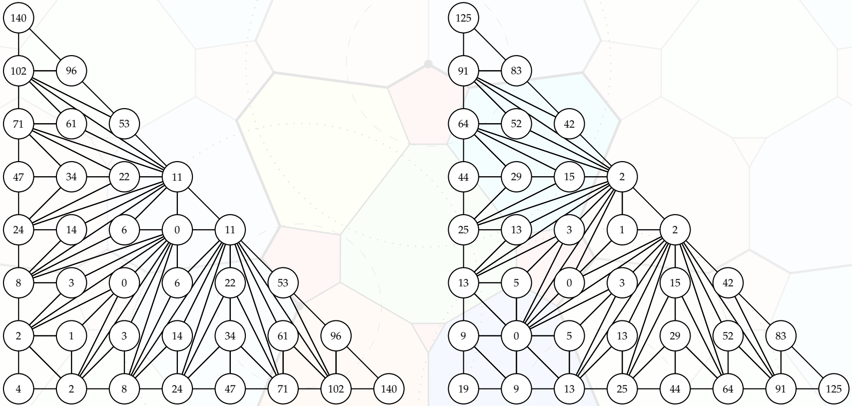



(a) bat (realizes 53 types) (b) moth (realizes 44 types)

Figure: Two regular triangulations of $6 \cdot \Delta_2$ realizing all nonempty real schemes types of degree six. Values at the vertices indicate lifting functions.

# Four Regular Triangulations to rule them all

## Characterization of curves of degree seven

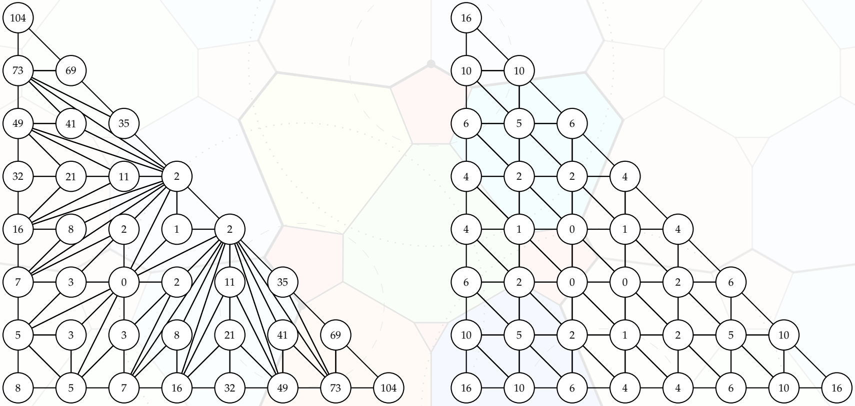



(a) cen (realizes 115 types) (b) spl (realizes 107 types)

Figure: The four regular triangulations of $7 \cdot \Delta_2$ realizing all real schemes types of degree seven. The values at the vertices indicate the respective lifting functions.

# Four Regular Triangulations to rule them all

## Characterization of curves of degree seven




(c) fra (realizes 103 types) (d) hon (realizes 47 types)

Figure: The four regular triangulations of $7 \cdot \Delta_2$ realizing all real schemes types of degree seven. The values at the vertices indicate the respective lifting functions.

# Final Remarks

1. AI can be used in various ways in modern mathematical research workflows (actual discovery, verification, etc); beyond simple black-box prompting
2. Fully-automatic discovery of new mathematics might be possible in the future but relies on *strong* verification approaches (LEAN might be too inefficient)
3. The agentic harness seems to be key: how does the agent receive feedback on its work, how is it guided, and which tools are available?
4. Empirically: the human-in-the-loop is crucial to guide the search

**The promises of AI4MATH are great** but need to go beyond simple black-box prompting "your favorite Erdős problem" (which then turns out having a solution that is already known)...

**Thank you!**

Mundinger, K., Pokutta, S., Spiegel, C., and Zimmer, M. (2024). Extending the Continuum of Six-Colorings. Geombinatorics Quarterly. Available at https://arxiv.org/abs/2404.05509.

Mundinger, K., Zimmer, M., Kiem, A., Spiegel, C., and Pokutta, S. (2025). Neural Discovery in Mathematics: Do Machines Dream of Colored Planes? Proceedings of the 42nd International Conference on Machine Learning (ICML), 267, 45236–45255. Available at https://arxiv.org/abs/2501.18527.

Geiselmann, Z., Joswig, M., Kastner, L., Mundinger, K., Pokutta, S., Spiegel, C., Wack, M., and Zimmer, M. (2026). 121 Patchworked Curves of Degree Seven. Preprint. Available at https://arxiv.org/abs/2602.06888.

# References I


K. I. Appel and W. Haken. Every planar map is four-colorable. *Illinois Journal of Mathematics*, 21(3):429–490, 1977. Computer-assisted proof of the four color theorem (first major such proof).

N. d. Bruijn and P. Erdos. A colour problem for infinite graphs and a problem in the theory of relations. *Indigationes Mathematicae*, 13:371–373, 1951.

A. D. De Grey. The chromatic number of the plane is at least 5. *arXiv preprint arXiv:1804.02385*, 2018.

G. Exoo and D. Ismailescu. The chromatic number of the plane is at least 5: a new proof. *Discrete & Computational Geometry*, 64(1):216–226, 2020.

Z. Geiselmann, M. Joswig, L. Kastner, K. Mundinger, S. Pokutta, C. Spiegel, M. Wack, and M. Zimmer. Patchworked Curves of Degree Seven. *preprint*, 2 2026. doi: 10.48550/arXiv.2602.06888.

D. A. Gudkov. The topology of real projective algebraic varieties. *Russ. Math. Surv.*, 29(4):1–79, 1974. ISSN 0036-0279. doi: 10.1070/RM1974v029n04ABEH001288.

J. Haase and S. Pokutta. Human–AI Cocreativity: Exploring synergies across levels of creative collaboration. In J. C. Kaufman and M. Worwood, editors, *Generative Artificial Intelligence and Creativity*, chapter 16, pages 205–221. 1 2026. doi: 10.1016/B978-0-443-34073-4.00009-5.

T. C. Hales, M. Adams, G. Bauer, D. T. Dang, J. Harrison, H. L. Truong, C. Kaliszyk, V. Magron, S. McLaughlin, N. T. Thang, N. Q. Truong, T. Nipkow, S. Obua, J. Pleso, J. Rute, A. Solovyev, et al. A formal proof of the kepler conjecture. *Forum of Mathematics, Pi*, 5:e2, 2017. doi: 10.1017/fmp.2017.1. Flyspeck project: complete formal verification of the Kepler conjecture.

M. J. H. Heule, O. Kullmann, and V. W. Marek. Solving and verifying the boolean pythagorean triples problem via cube-and-conquer. In *Theory and Applications of Satisfiability Testing – SAT 2016*, volume 9710 of *Lecture Notes in Computer Science*, pages 228–245. Springer, 2016. Computer-intensive proof generating a multi-terabyte certificate.

D. Hilbert. Mathematische Probleme. *Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl.*, 1900:253–297, 1900. URL `http://eudml.org/doc/58479`. English translation (M. F. Winston Newson): Bull. Amer. Math. Soc. 8 (1902), 437–479.

I. Hoffman and A. Soifer. Almost chromatic number of the plane. *Geombinatorics*, 3(2):38–40, 1993.

I. Hoffman and A. Soifer. Another six-coloring of the plane. *Discrete Mathematics*, 150(1-3):427–429, 1996.

I. Itenberg and O. Viro. Patchworking algebraic curves disproves the Ragsdale conjecture. *Math. Intelligencer*, 18(4):19–28, 1996. ISSN 0343-6993,1866-7414. doi: 10.1007/BF03026748.

D. Mixon. Polymath16, seventeenth thread: Declaring victory. *Polymath16*, February 1 2021. Retrieved 16 August 2021.

L. Moser and M. Moser. Solution to problem 10. *Canadian Mathematical Bulletin*, 4:187–189, 1961.

K. Mundinger, S. Pokutta, C. Spiegel, and M. Zimmer. Extending the Continuum of Six-Colorings. *Geombinatorics Quarterly*, 5 2024.

K. Mundinger, M. Zimmer, A. Kiem, C. Spiegel, and S. Pokutta. Neural Discovery in Mathematics: Do Machines Dream of Colored Planes? *Proceedings of the 42nd International Conference on Machine Learning (ICML)*, 267:45236–45255, 5 2025. ISSN 2640-3498. doi: 10.48550/arXiv.2501.18527.

J. F. Nash and M. T. Rassias. *Open problems in mathematics*. Springer, 2016.

# References II


V. V. Nikulin. Integer symmetric bilinear forms and some of their geometric applications. *Izv. Math.*, 14(1):103–167, 1980.

S. Y. Orevkov. Classification of flexible $M$-curves of degree 8 up to isotopy. *Geom. Funct. Anal.*, 12(4):723–755, 2002. ISSN 1016-443X,1420-8970. doi: 10.1007/s00039-002-8264-6. URL `https://doi.org/10.1007/s00039-002-8264-6`.

S. Y. Orevkov. Complex orientation formulas for $M$-curves of degree $4d+1$ with 4 nests. *Ann. Fac. Sci. Toulouse Math. (6)*, 19(1):13–26, 2010. ISSN 0240-2963,2258-7519. URL `http://afst.cedram.org/item?id=AFST_2010_6_19_1_13_0`.

J. Parts. What percent of the plane can be properly 5-and 6-colored? *arXiv preprint arXiv:2010.12668*, 2020.

D. Pritikin. All unit-distance graphs of order 6197 are 6-colorable. *Journal of Combinatorial Theory, Series B*, 73(2):159–163, 1998.

N. Robertson, D. P. Sanders, P. Seymour, and R. Thomas. The four colour theorem. *Journal of Combinatorial Theory, Series B*, 70(1):2–44, 1997. Shorter and fully computer-verified proof reducing the number of configurations.

V. A. Rohlin. Complex topological characteristics of real algebraic curves. *Uspekhi Mat. Nauk*, 33(5(203)):77–89, 237, 1978. ISSN 0042-1316.

A. Soifer. A six-coloring of the plane. *Journal of Combinatorial Theory, Series A*, 61(2):292–294, 1992.

A. Soifer. Six-realizable set $x_6$. *Geombinatorics*, III(4):140–145, 1994a.

A. Soifer. An infinite class of six-colorings of the plane. *Congressus Numerantium*, pages 83–86, 1994b.

A. Soifer. *The mathematical coloring book: Mathematics of coloring and the colorful life of its creators*. Springer, 2009.

A. Soifer. *The New Mathematical Coloring Book*. Springer, 2024.

O. Y. Viro. Gluing of plane real algebraic curves and constructions of curves of degrees 6 and 7. In *Topology (Leningrad, 1982)*, volume 1060 of *Lecture Notes in Math.*, pages 187–200. Springer, Berlin, 1984. ISBN 3-540-13337-2. doi: 10.1007/BFb0099934. URL `https://doi.org/10.1007/BFb0099934`.

O. Y. Viro. Progress in the topology of real algebraic varieties in the last six years. *Uspekhi Mat. Nauk*, 41(3(249)):45–67, 240, 1986. ISSN 0042-1316.