

# Neural Discovery in Mathematics: Do Machines Dream of Colored Planes?

Sebastian Pokutta

Technische Universität Berlin  
and  
Zuse Institute Berlin

[pokutta@math.tu-berlin.de](mailto:pokutta@math.tu-berlin.de)  
@spokutta

Baptist University

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ZUSE  
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BERLIN



Berlin Mathematics Research Center

**MATH+**

# What is this talk about?

## Introduction

*What AI can do for Mathematics? Trends and a concrete example.*

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**Today:** Brief overview of approach, two new constructions, and outlook.

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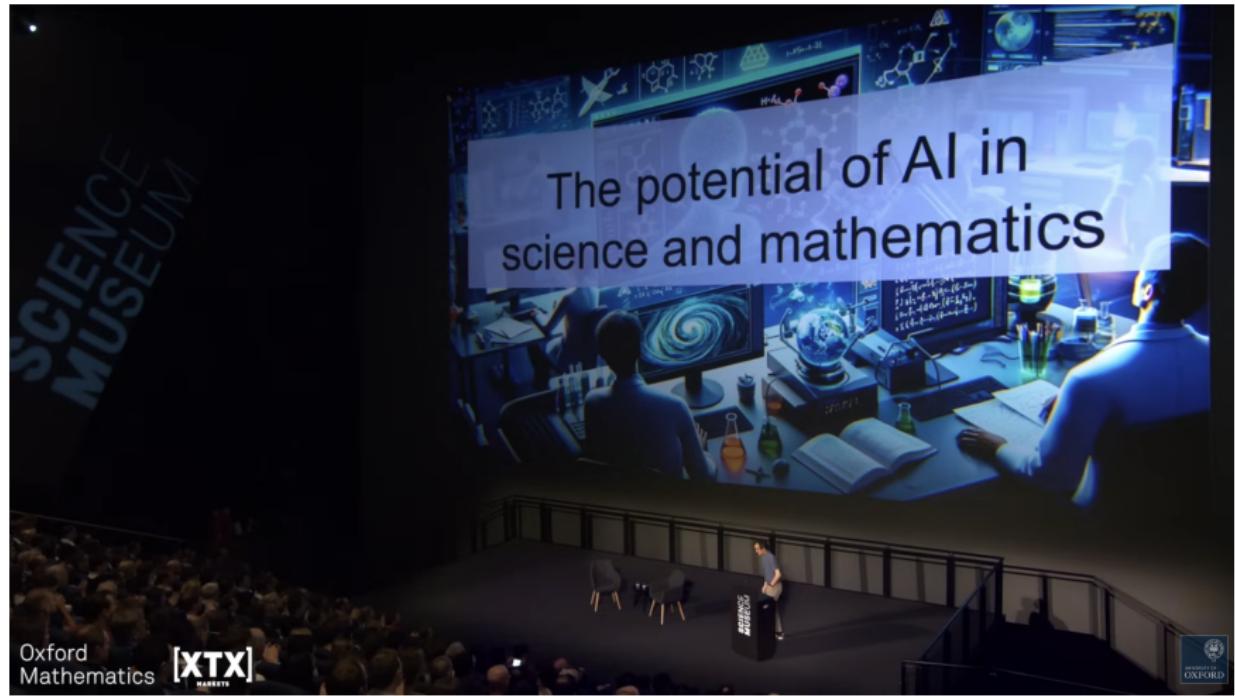
## Outline

- AI and Mathematics
- The Hadwiger-Nelson Problem: formulation and status
- Machine learning approach
- Results a.k.a. constructions

(Hyperlinked) References are not exhaustive; check references contained therein.

# What is this talk about?

AI and Mathematics



[The Potential for AI in Science and Mathematics - Terence Tao]

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## AI and Mathematics

### Various levels of co-creation.

- **Digital Pen**: basically like autocorrect, bibtex lookup, etc.
- **AI Task Specialist**: ChatGPT, Claude, Gemini, etc.
- **AI Assistant**: Agents with integrated tools, verification, etc.
- **AI Co-creator**: Fully integrated, autonomous, co-creator

[Haase and Pokutta, 2026]

"2000s"

2022 - 2025

2025 - present

2027(??)

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Capabilities are **impressive** but **unstable**:

- SOTA models achieve post-PhD level scores on benchmarks, yet in day-to-day use make trivial (logical) errors.
- No hard verification of results and randomness across runs.
- Prompting and scaffolding are still a challenge.
- Availability of tools for verification etc crucial.

# What is this talk about?

Mathematics with computers is not new

## Various high-profile examples from the past.

- Four Color Theorem: massive computer-based case checking

[Appel and Haken, 1977, Robertson et al., 1997]

- Kepler Conjecture / Hales' Theorem: extensive computer verification

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- Classification of Finite Simple Groups: Formal verification with Lean and Coq

- Boolean Pythagorean Triples Problem: A spectacular 200TB SAT-solver proof.

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## Crucial role in computational mathematics / scientific computing

- Finite Elements
- Numerical Simulations
- Optimization
- Engineering
- ...

## The Hadwiger-Nelson Problem

joint work with: Aldo Kiem, Konrad Mundinger,  
Christoph Spiegel, Max Zimmer

ICML 2025 (oral)

<https://arxiv.org/abs/2404.05509>

# The Hadwiger-Nelson Problem

Problem (Nelson 1950, also: Gardner, Moser, Erdős, Harary, Tutte, ...)

*What is the smallest number of colors sufficient for coloring the plane in such a way that no two points of the same color are at a unit distance apart?*



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Infinite graph with vertex set  $\mathbb{E}^2$  and edges  $\{x, y\}$  for any  $x, y \in \mathbb{E}^2$  with  $\|x - y\| = 1$   
 $\Rightarrow$  chromatic number of the plane  $\chi(\mathbb{E}^2)$

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## Theorem

Assuming Axiom of Choice (AoC):

*Any graph is  $k$ -colorable iff every finite subgraph of it is  $k$ -colorable.*

[Bruijn and Erdos, 1951]

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This problem has a long and complicated history...

over 14 pages in [Soifer, 2024]

# History

## The Hadwiger-Nelson Problem

**Table 3.1** Who created the chromatic number of the plane problem?

Publication	Year	Author(s)	Problem creator(s) or source named
[Gar2]	1960	Gardner	<b>“Leo Moser ...writes...”</b>
[Had4]	1961	Hadwiger (after Klee)	<b>Nelson</b>
[E61.22]	1961	Erdős	“I cannot trace the origin of this problem”
[Cro]	1967	Croft	“A long <sup>18</sup> -standing open problem of <b>Erdős</b> ”
[Woo1]	1973	Woodall	<b>Gardner</b>
[Sim]	1976	Simmons	<b>Erdős, Harary, and Tutte</b>
[E80.38]	1980–	Erdős	<b>Hadwiger and Nelson</b>
[E81.23]	1981		
[E81.26]			
[CFG]	1991	Croft, Falconer, and Guy	“Apparently due to <b>E. Nelson</b> ”
[KW]	1991	Klee and Wagon	“Posed in 1960–61 by <b>M. Gardner</b> and <b>Hadwiger</b> ”

p. 24 in [Soifer, 2024]



**Lower bounds on  $\chi(\mathbb{E}^2)$**

# Lower bounds through unit distance graphs

The Hadwiger-Nelson Problem

Find unit distance graphs of large chromatic number.

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### Definition

A graph  $G = (V, E)$  is a **unit distance graph** if there exists an embedding  $f : V \rightarrow \mathbb{E}^2$  of its vertices in the plane s.t.  $\|f(u) - f(v)\| = 1$  if and only  $\{u, v\} \in E$ .

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[Moser and Moser, 1961]

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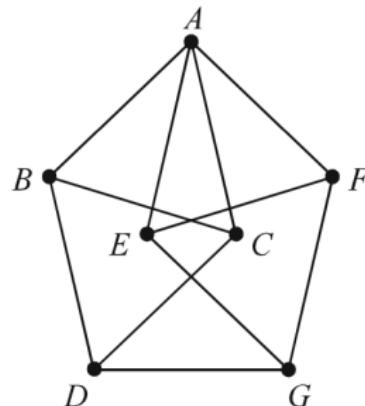
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### Theorem

*There is a unit distance graph on 20 425 vertices with chromatic number 5.*

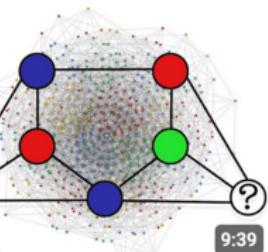
[De Grey, 2018]

# Lower bounds through unit distance graphs

## The Hadwiger-Nelson Problem

Find unit distance graphs of large chromatic number.

Numberphile



9:39

### A Colorful Unsolved Problem - Numberphile

681K views • 5 years ago



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More links & stuff in full description below ↓↓

Numberphile is supported by the Mathematical Science...

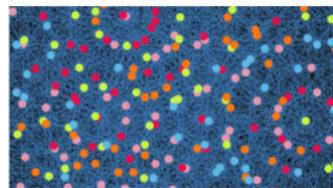
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# Lower bounds through unit distance graphs

## The Hadwiger-Nelson Problem

Find unit distance graphs of large chromatic number.



GRAPH THEORY

### Decades-Old Graph Problem Yields to Amateur Mathematician

By EVELYN LAMB | APRIL 17, 2018 | 26

...number of vertices? The problem, now known as the Hadwiger-Nelson problem or the problem of finding the chromatic number of the plane, has piqued the interest of many mathematicians, including...



# Lower bounds through unit distance graphs

## The Hadwiger-Nelson Problem

Find unit distance graphs of large chromatic number.



Aubrey de Grey and Alexander Soifer, *Il Vicino*, January 18, 2020



Ronald L. Graham presents Aubrey D.N.J. de Grey the Prize: \$1000, San Diego, September 22, 2018

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[De Grey, 2018]

Simpler constructions with...

1. 1581 vertices
2. 627 vertices
3. 553 vertices (as part of Polymath16)
4. 509 vertices (as part of Polymath16)

for detail see [De Grey, 2018]

[Exoo and Ismailescu, 2020]

Marijn Heule, for details see [Mixon, 2021]

Jaan Parts, for details see [Mixon, 2021]



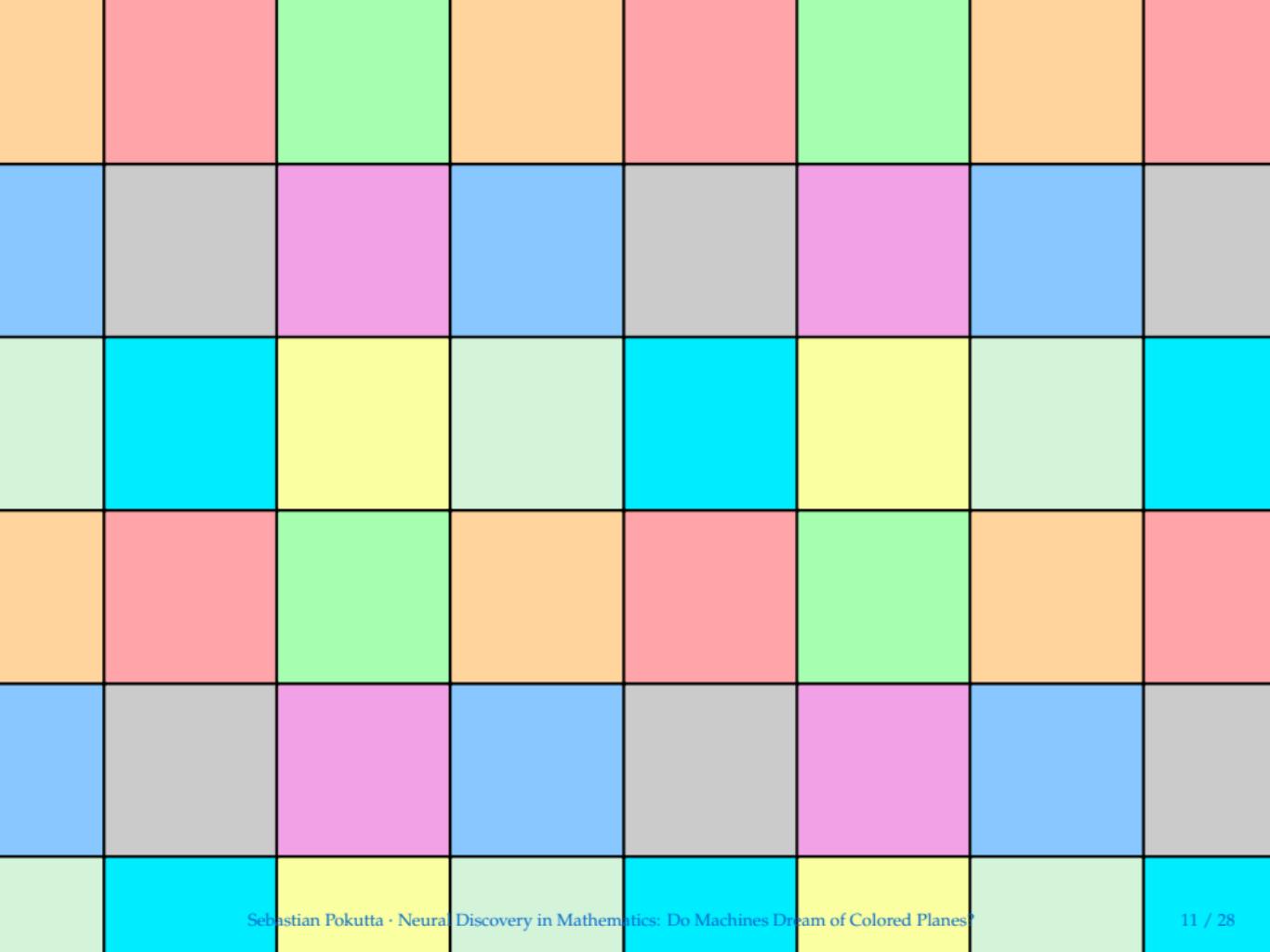
Upper bounds on  $\chi(\mathbb{E}^2)$

# Upper bounds through colorings

## The Hadwiger-Nelson Problem

Explicit colorings  $g : \mathbb{E}^2 \rightarrow [c] := \{1, \dots, c\}$ , usually derived through tessellations using simple polytopal shapes, give

$$5 \leq \chi(\mathbb{E}^2) \leq \dots$$

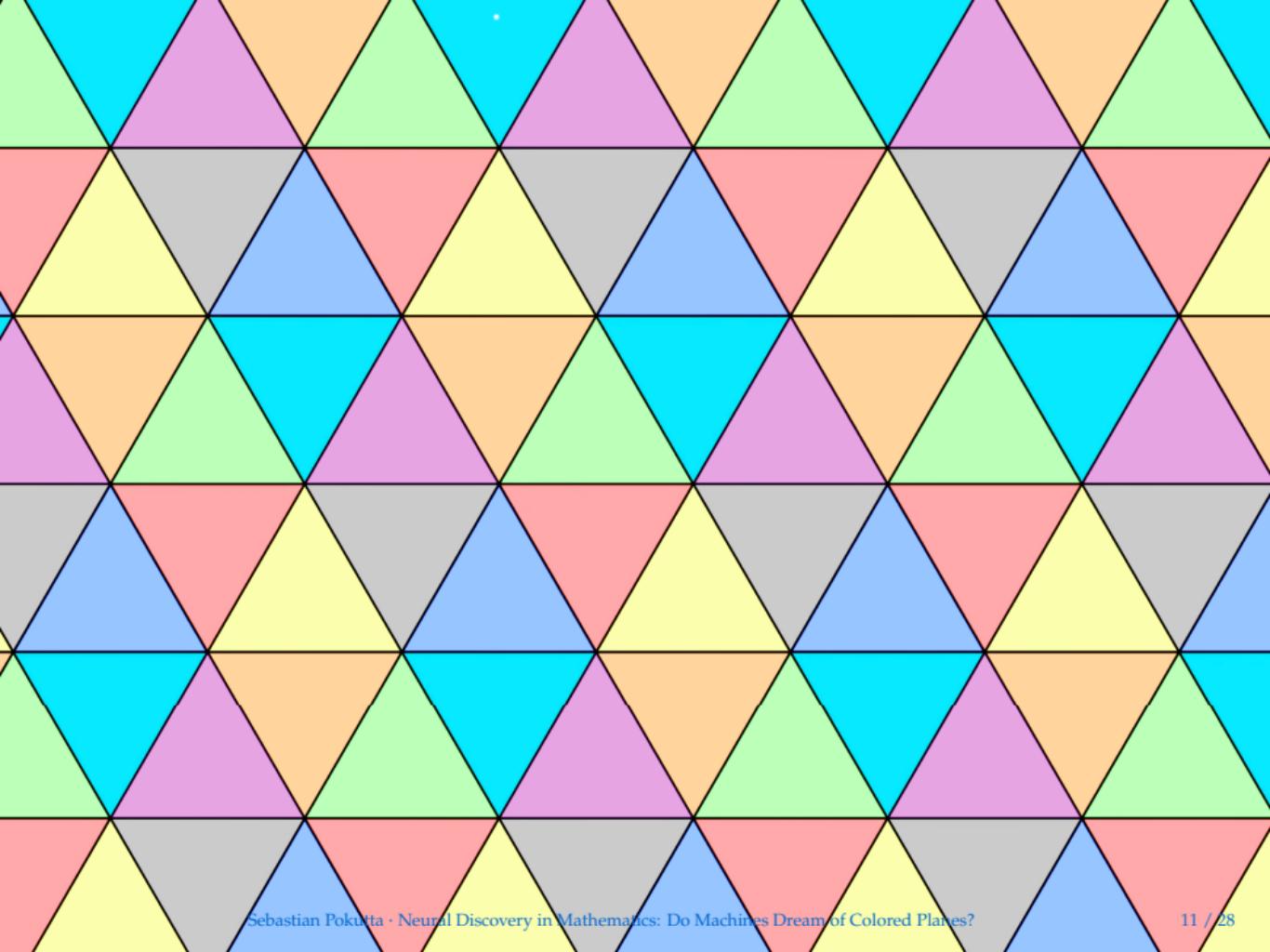


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$$5 \leq \chi(\mathbb{E}^2) \leq 9.$$

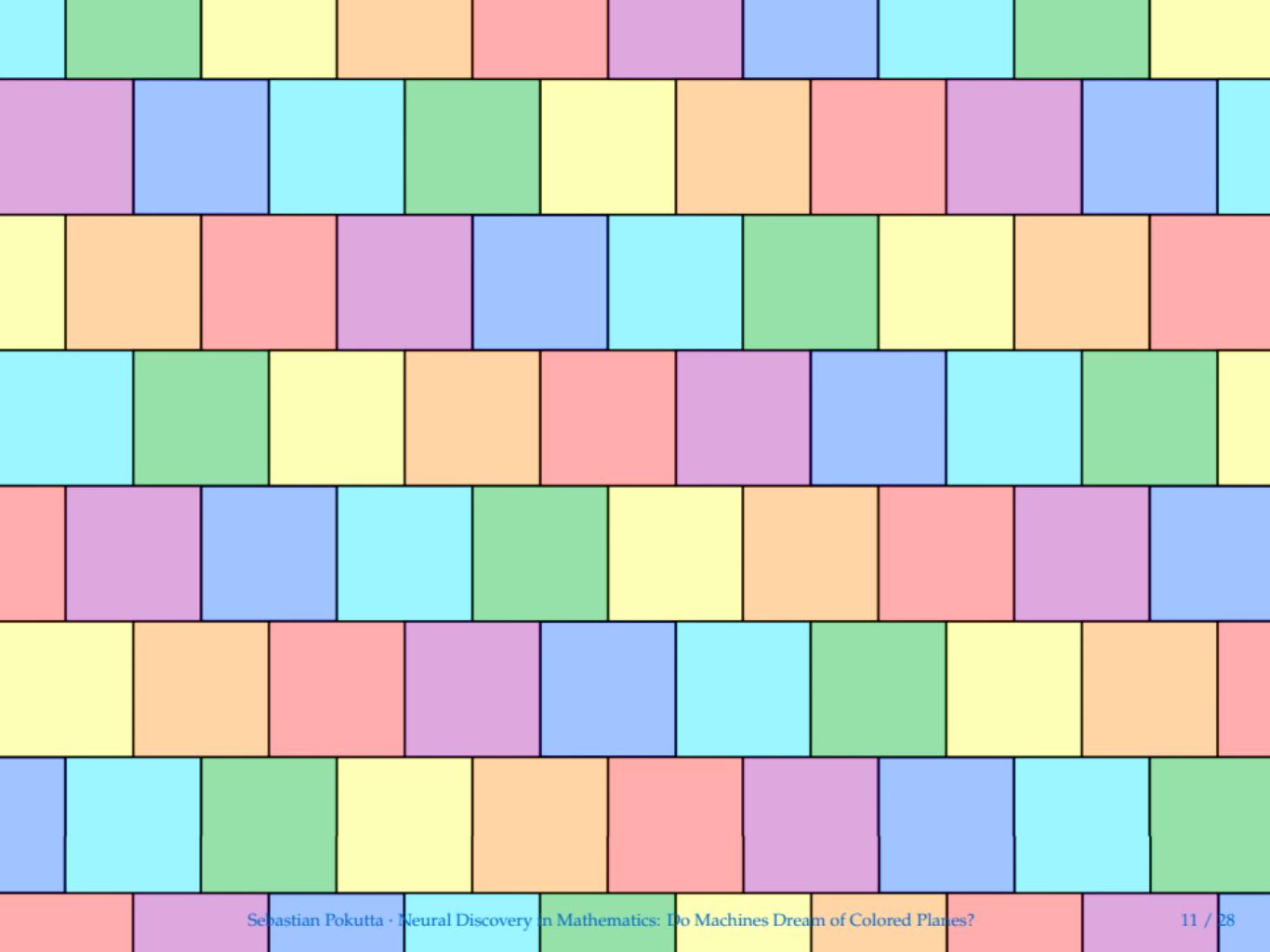


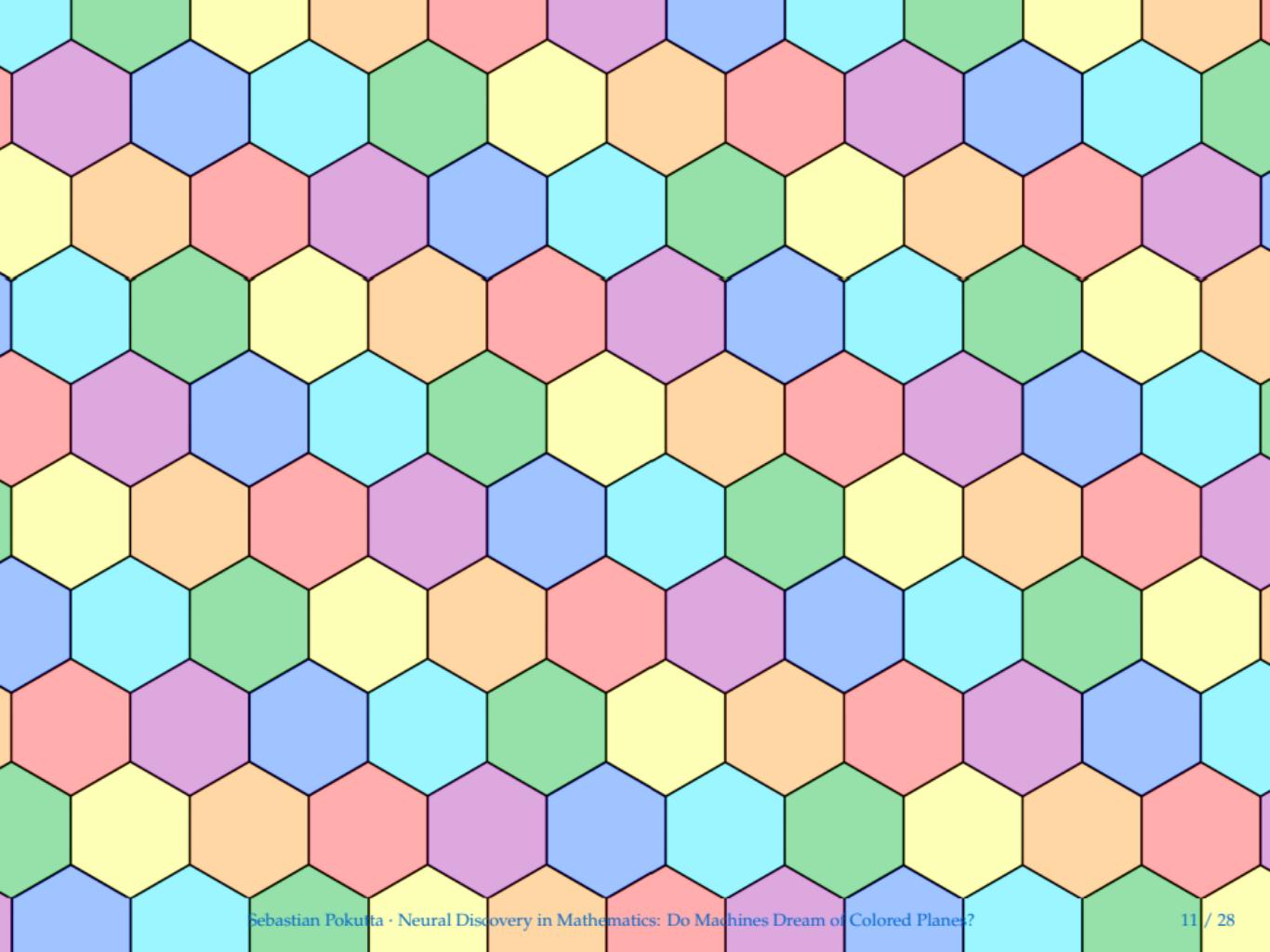
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$$5 \leq \chi(\mathbb{E}^2) \leq 8.$$





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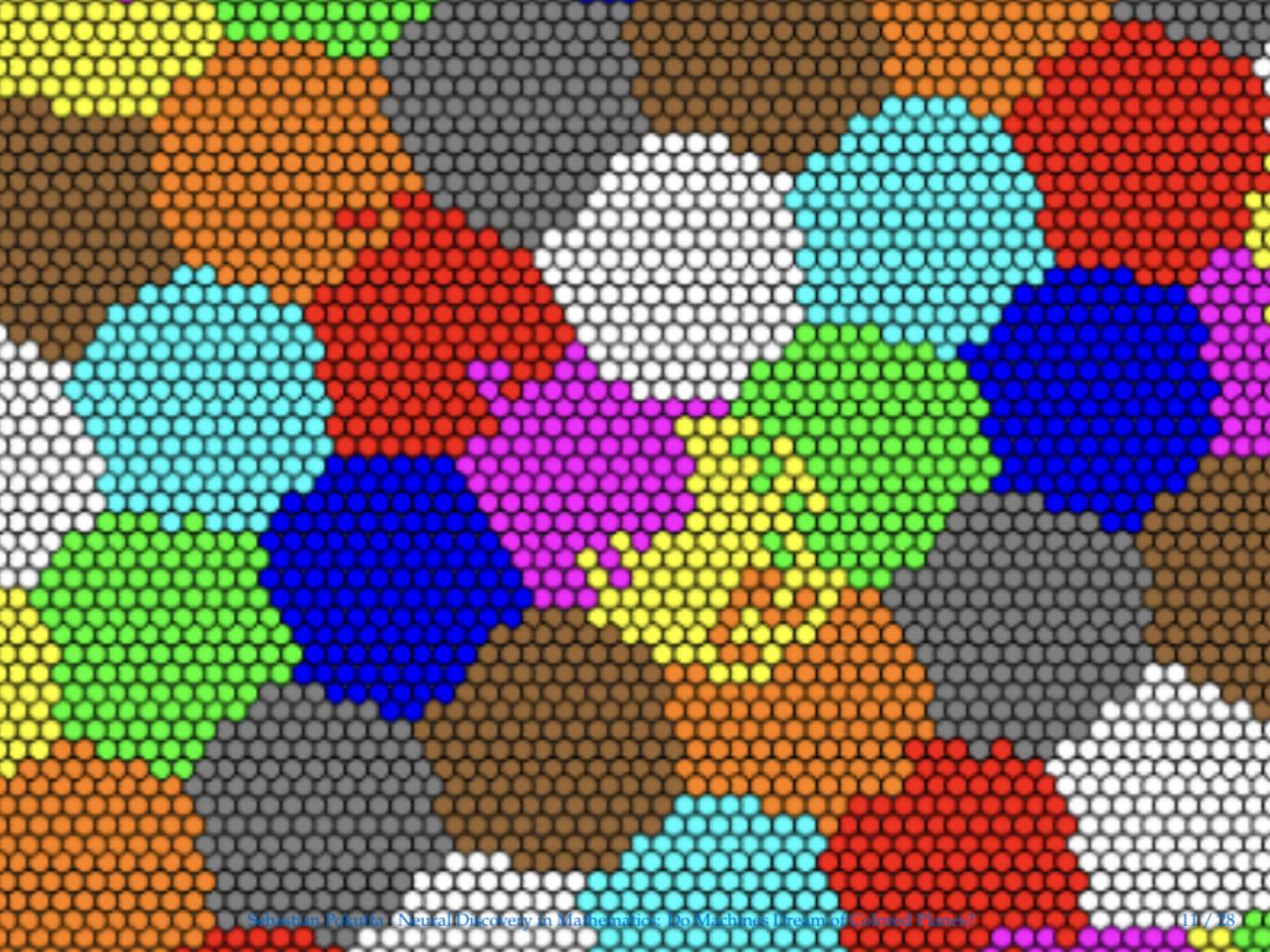
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**Question.** Can we use computers to find admissible colorings  $g : \mathbb{E}^2 \rightarrow [c]$ , i.e.,

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... attempts, e.g., via discretization and SAT solvers...



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**Idea.** Use a parameterized and easily differentiable family  $g_\theta : \mathbb{E}^2 \rightarrow \Delta_c$  and find

$$\arg \min_{\theta} \mathbb{E} \left[ \int_{B_1(x)} g_\theta(x) \cdot g_\theta(y) dy \mid x \in \mathbb{E}^2 \right].$$

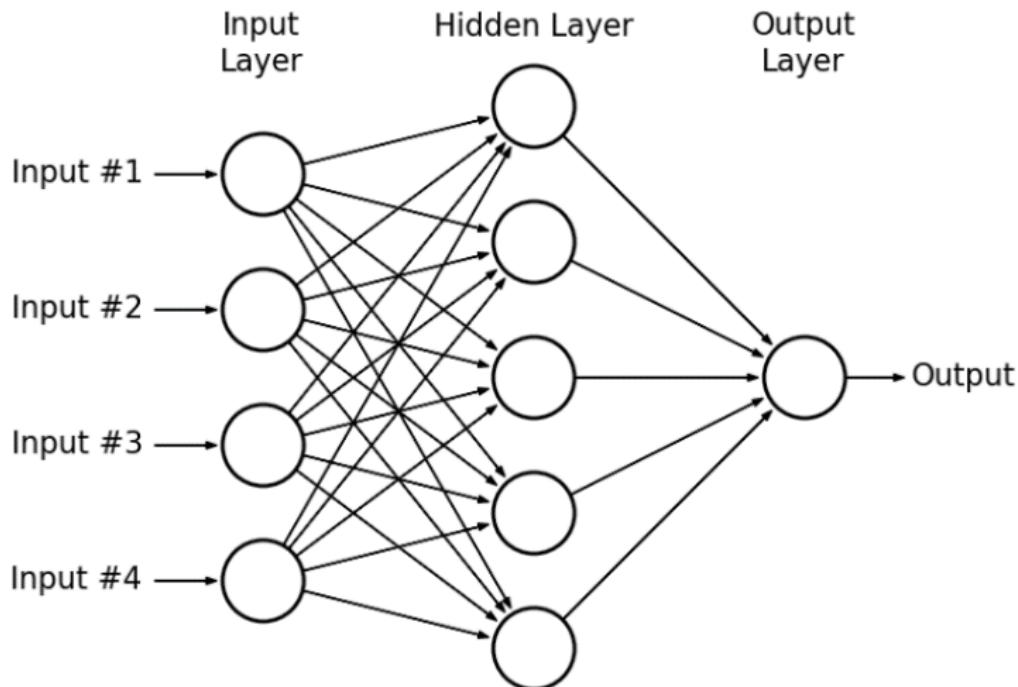
**Key Point.** Approach is continuous in nature.



New upper bounds via  
machine learning?

# One second recap: Neural Networks

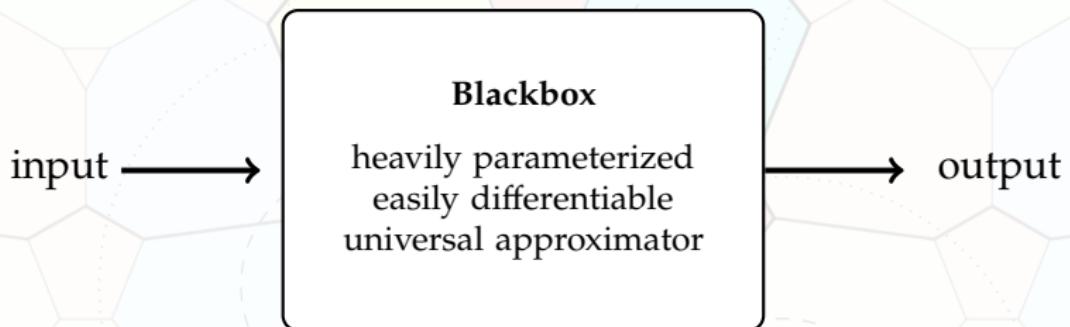
## Neural Networks as Colorings



# One second recap: Neural Networks

## Neural Networks as Colorings

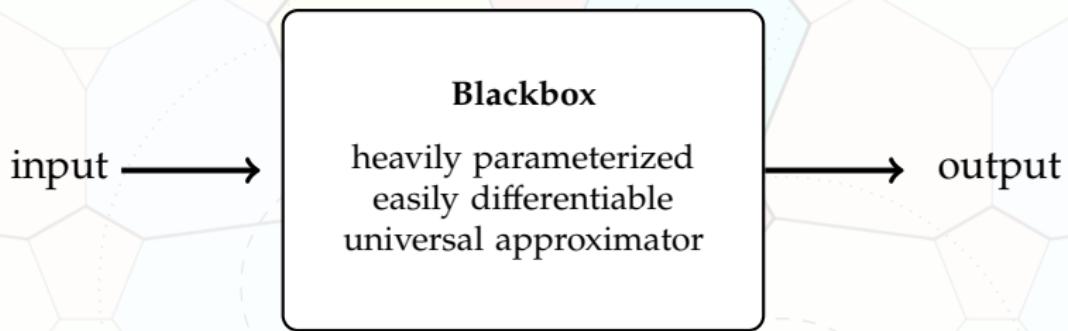
**Here.** Simply a parameterized continuous function to model the coloring.



# One second recap: Neural Networks

## Neural Networks as Colorings

**Here.** Simply a parameterized continuous function to model the coloring.



### Theorem (Universal Approximation Theorem)

*Feedforward neural networks with certain activation functions are dense (w.r.t. compact convergence) in the space of continuous functions.*

# Can we improve the upper bound?

## Neural Networks as Colorings

**Idea.** Use gradient descent to train a feedforward network  $g_\theta$  to minimize

$$L(\theta) = \int_{[-b,b] \times [-b,b]} \int_{B_1(x)} g_\theta(x) \cdot g_\theta(y) dy dx$$

for some reasonable  $b \in \mathbb{R}$ ?

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where  $\nabla_\theta g_\theta(x^{(i)}) \cdot g_\theta(y^{(i)})$  is easily computed through backpropagation, to adjust the parameters  $\theta$  with an appropriate step size  $\alpha_k$  through

$$\theta_{k+1} = \theta_k - \alpha_k \hat{\nabla}_\theta L(\theta).$$

⇒ **Very flexible approach “Deep Annealing”**

(also: tropicalization of loss function aka softmax... “minimize the max”)



# Unfortunately this coloring was already known...

## Neural Networks as Colorings

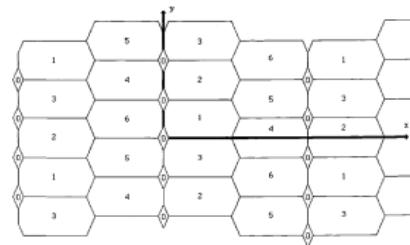
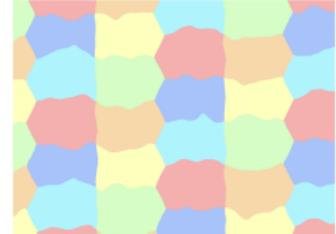
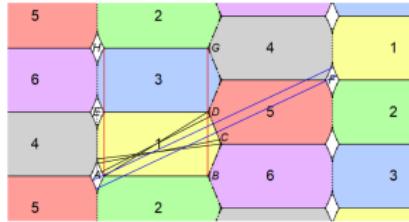


FIG. 3. A good 7-coloring of  $(\mathbb{R}^2, 1)$ .



## Theorem

99.985% of the plane can be colored with 6 colors such that no two points of the same color are a unit distance apart.

[Pritikin, 1998, Parts, 2020]

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## Neural Networks as Colorings

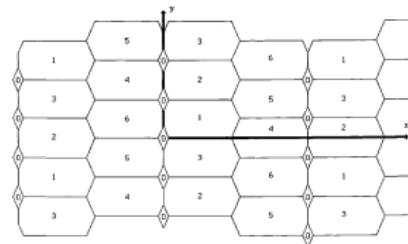
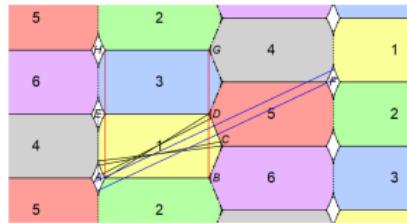


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## Corollary

*Any unit distance graph with chromatic number 7 must have at least 6 992 vertices.*

⇒ While coloring was known already maybe on the right track?



Off-diagonal variant

# Going off-diagonal

## Neural Networks as Colorings

If we cannot solve the original problem, we study variants of it:

# Going off-diagonal

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A  $c$ -coloring of the plane has **type**  $(d_1, \dots, d_c)$   
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**Problem (Soifer in Nash and Rassias' *Open Problems in Mathematics*)**

Determine the continuum of six-colorings  $X_6 = \{d \mid (1, 1, 1, 1, 1, d) \text{ can be realized}\}$ .

[Soifer, 1994a, Nash and Rassias, 2016]

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**Status.** Six-colorings exist for:

1.  $d = 1/\sqrt{5}$
2.  $d = \sqrt{2} - 1$
3. Part of family with  $0.414 \approx \sqrt{2} - 1 \leq d \leq 1/\sqrt{5} \approx 0.447$

[Soifer, 1992]

[Hoffman and Soifer, 1993, 1996]

[Hoffman and Soifer, 1996, Soifer, 1994b, 2009]

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if color  $i$  does not contain any points at distance  $d_i$ .

**Problem (Soifer in Nash and Rassias' *Open Problems in Mathematics*)**

Determine the continuum of six-colorings  $X_6 = \{d \mid (1, 1, 1, 1, 1, d) \text{ can be realized}\}$ .

[Soifer, 1994a, Nash and Rassias, 2016]

**Status.** Six-colorings exist for:

1.  $d = 1/\sqrt{5}$
2.  $d = \sqrt{2} - 1$
3. Part of family with  $0.414 \approx \sqrt{2} - 1 \leq d \leq 1/\sqrt{5} \approx 0.447$

[Soifer, 1992]

[Hoffman and Soifer, 1993, 1996]

[Hoffman and Soifer, 1996, Soifer, 1994b, 2009]

**Deep Annealing approach provides two new colorings leading to...**

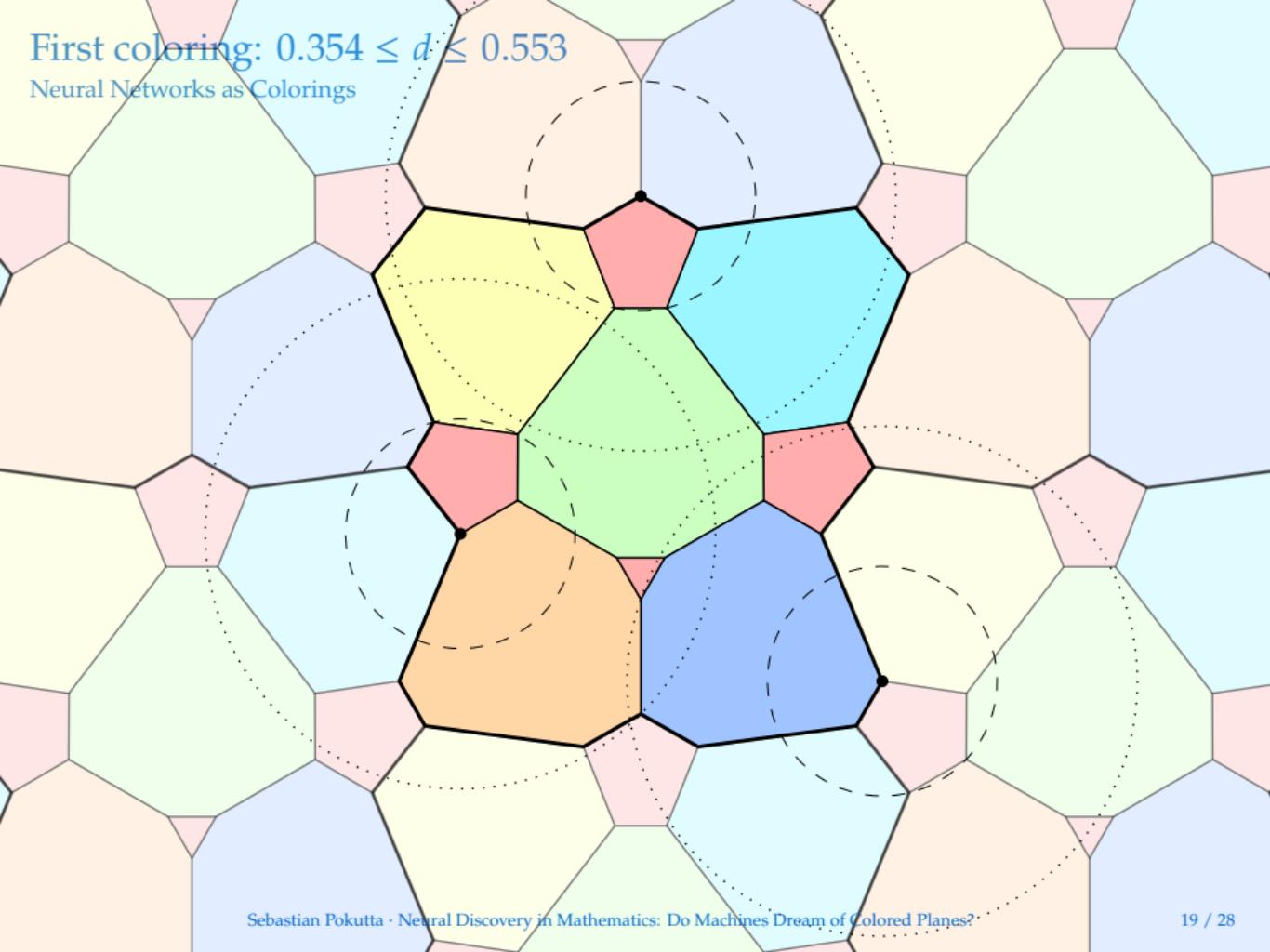
**Theorem**

$X_6$  contains the closed interval  $[0.354, 0.657]$ .

[Mundinger et al., 2024]

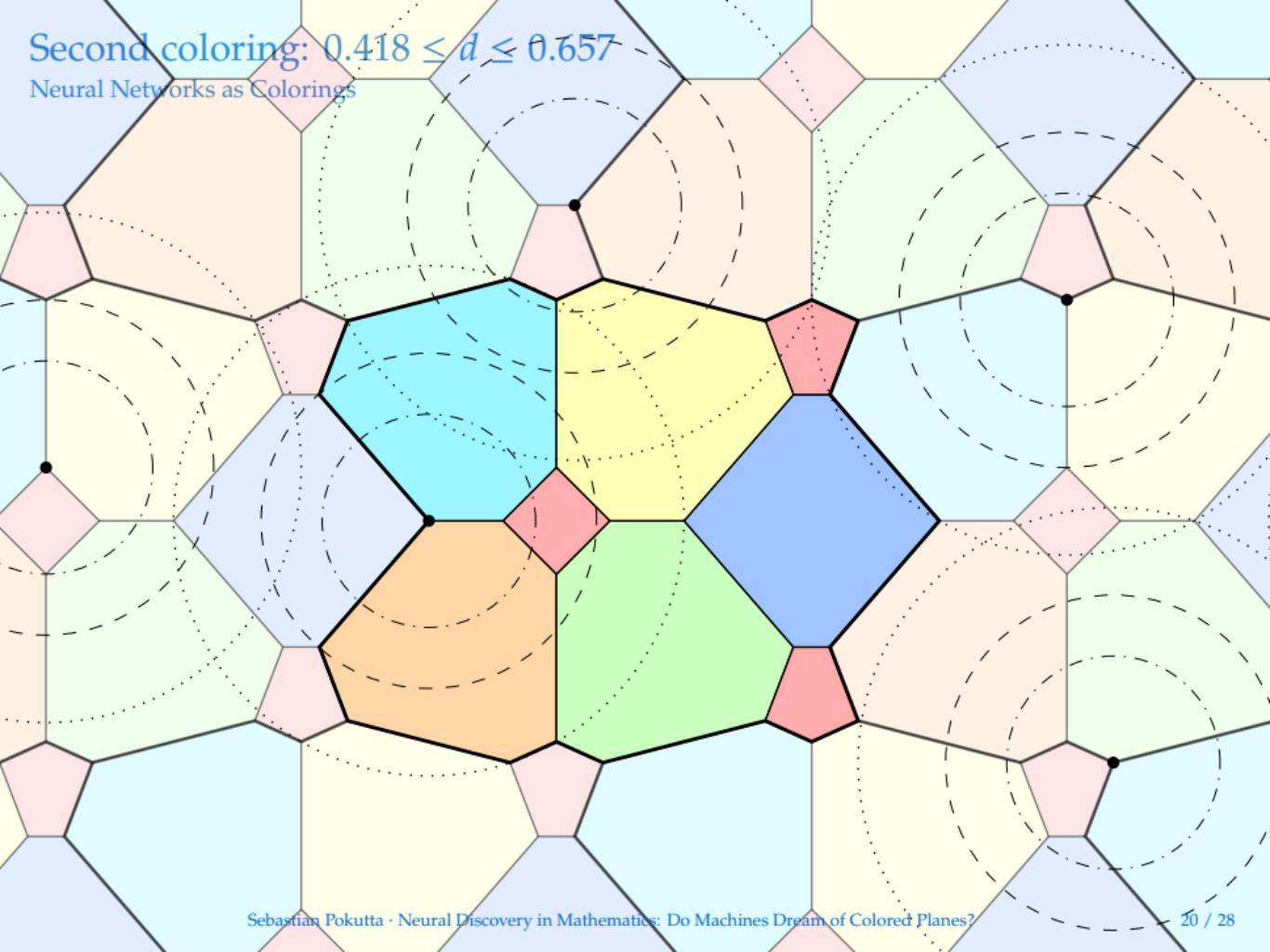
First coloring:  $0.354 \leq d \leq 0.553$

Neural Networks as Colorings



Second coloring:  $0.418 \leq d \leq 0.657$

Neural Networks as Colorings



# GEOMBINATORICS QUARTERLY

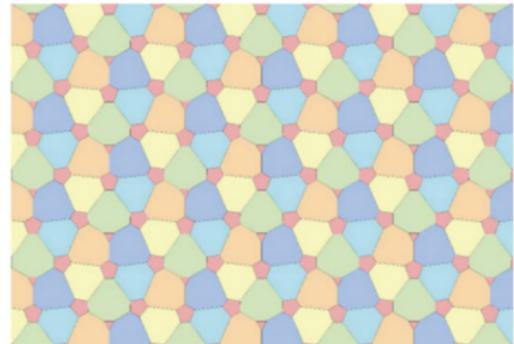
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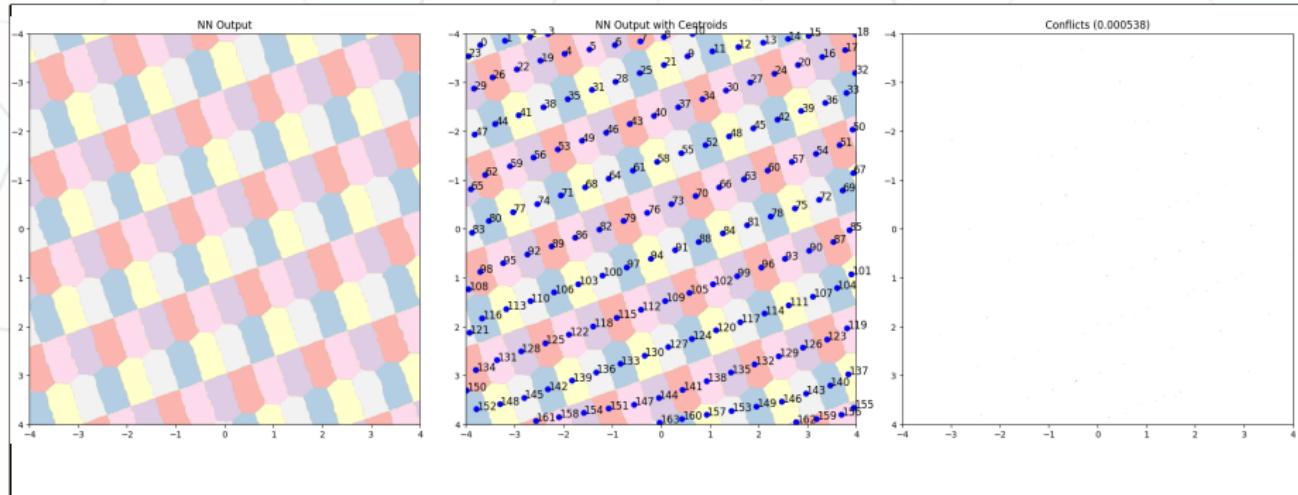
Volume XXXIV

October 2024

Issue 2

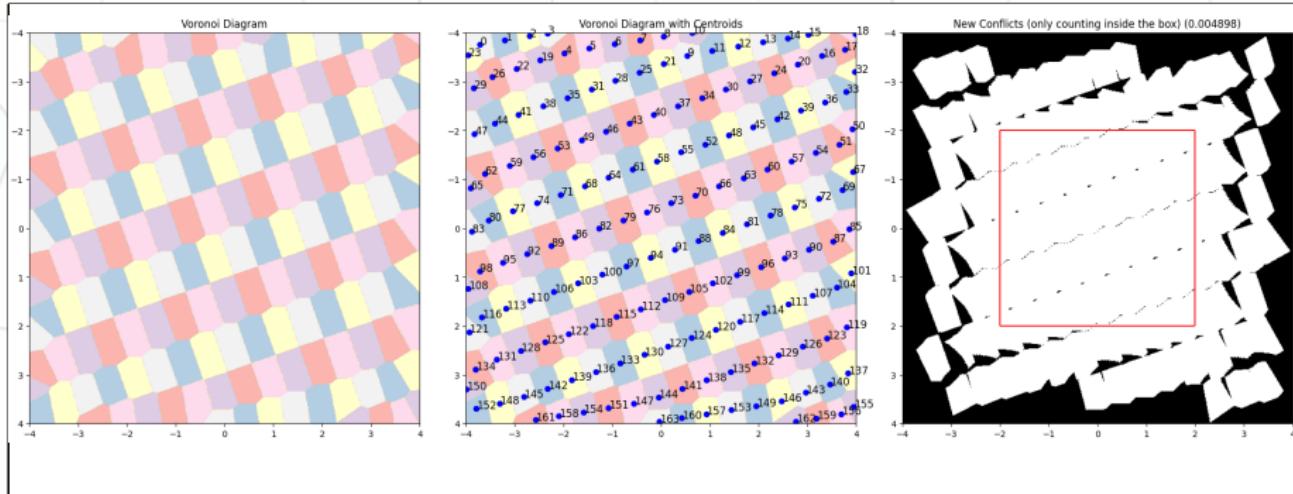
# Just numerics...?

## Neural Networks as Colorings



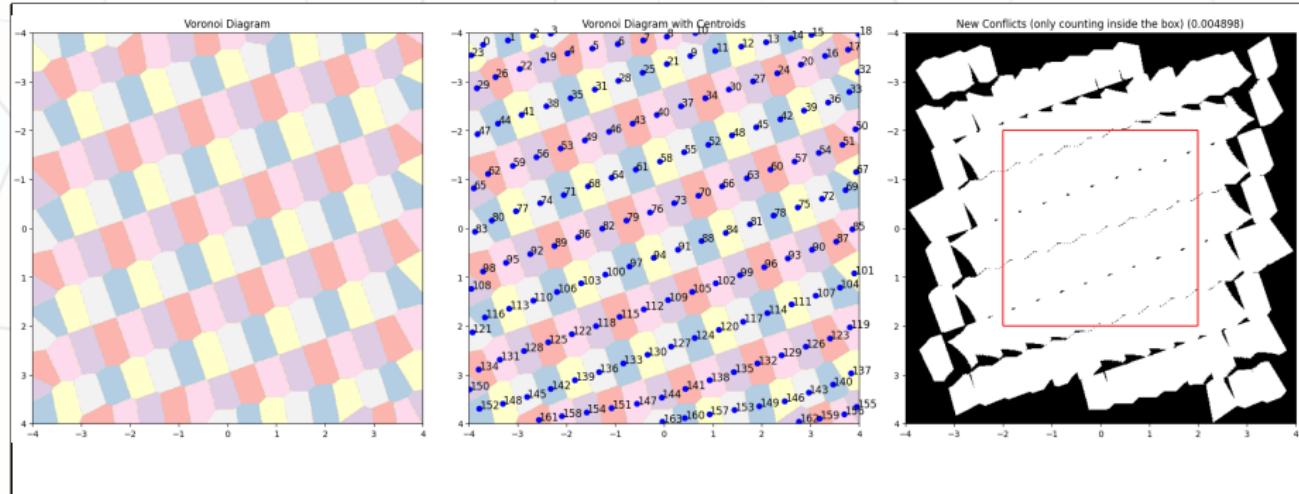
# Just numerics...?

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## Just numerics...?

## Neural Networks as Colorings

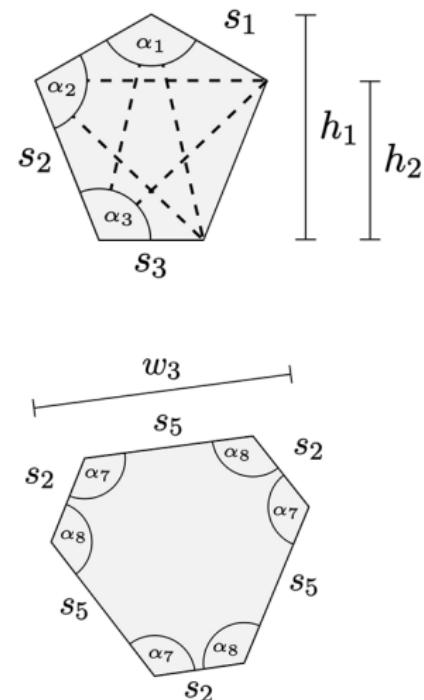
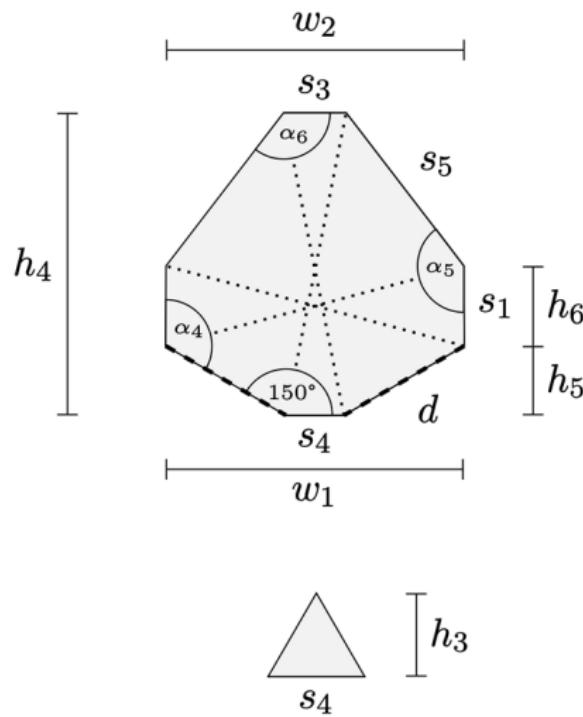


## Voronoi cell filtering...

⇒ Exact constructions for both colorings.

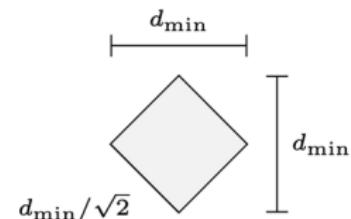
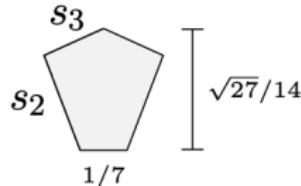
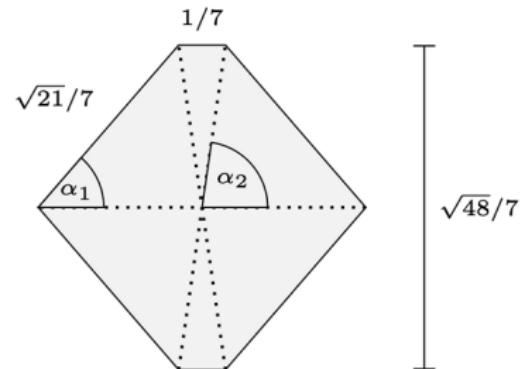
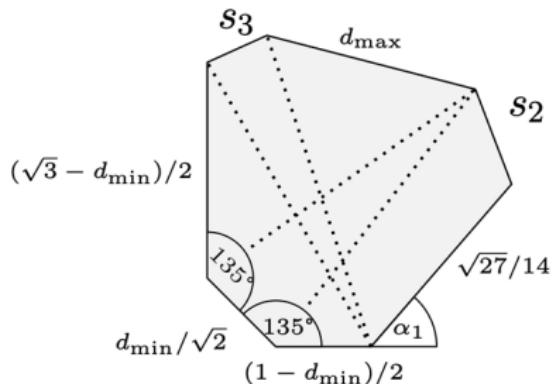
# First coloring: exact components

## Neural Networks as Colorings



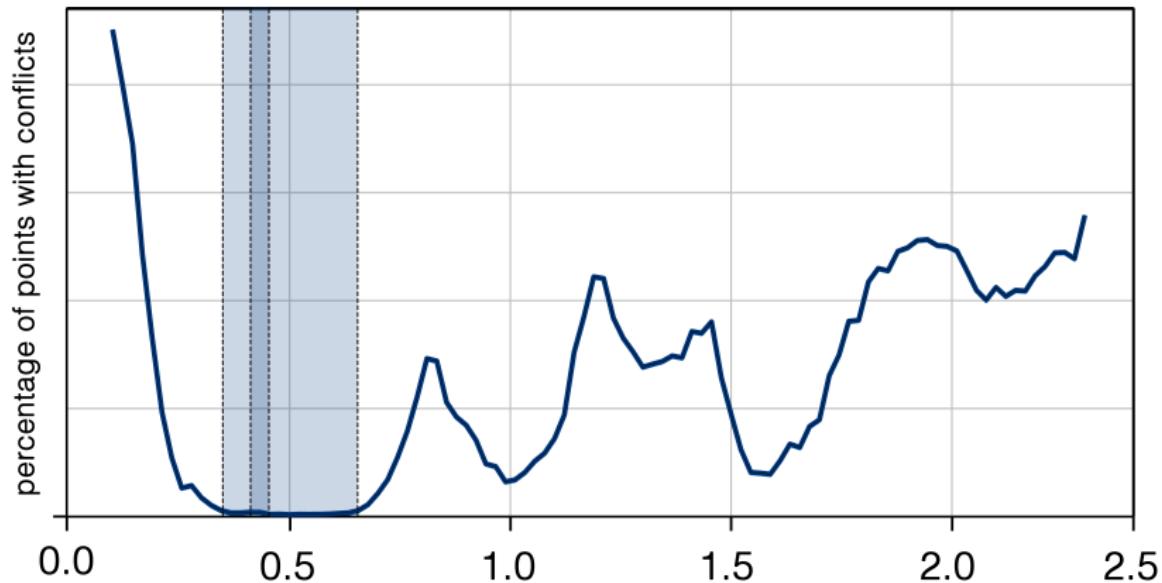
## Second coloring: exact components

### Neural Networks as Colorings



# Is this optimal?

## Neural Networks as Colorings



Numerical results showing the percentage of points with some conflict for a given forbidden distance  $d$  in the sixth color found over several runs.

# Under the hood

## Neural Networks as Colorings

### External Links.

1. From a bad trip to colorings...
2. Example of system output
3. Coloring 1 for different  $d$  in last color

## Open Problems and Final Remarks

The underlying optimization approach is very flexible:

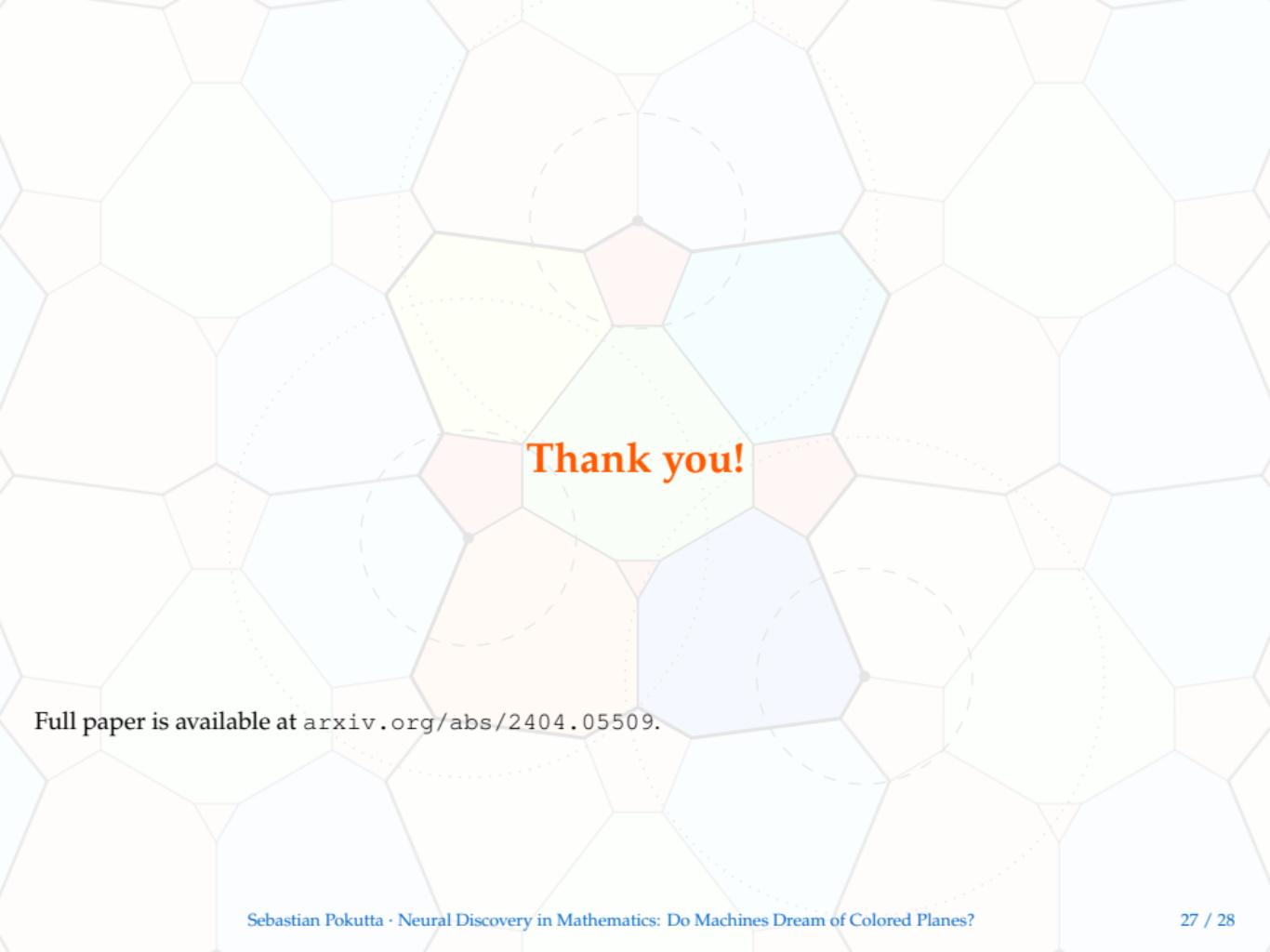
1. Can we improve the upper bound of the **polychromatic number** from 6 to 5?
2. Can we improve the upper bound of the **chromatic number of  $\mathbb{E}^3$**  from 15 to 14?
3. Can we apply the same ideas to generate **graphons** and other limit structures?
4. Can we use **adversarial networks** when the objective is non-differentiable?
5. Automatic **formal verification** of constructions with LEAN?

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**The key question.** Are we still working on the 6-coloring?



Thank you!

Full paper is available at [arxiv.org/abs/2404.05509](https://arxiv.org/abs/2404.05509).

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