Splitting algorithms via Linear Optimization Oracles

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Discrete Optimization. A conference in honor of Robert Weismantel

September 11th, 2025 · Zurich, Switzerland

Berlin Mathematics Research Center

What is this talk about?

Introduction

Given P, Q compact convex sets,
does there exist $x \in P \cap Q$?

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does there exist $x \in P \cap Q$?

Why? At the core of many algorithms. Allows for optimization via binary search.

Today. von Neumann's approach and a couple of new algorithms.

(Hyperlinked) References are not exhaustive; check references contained therein.

Some trivial insights...

Polytopes: H-representation and V-representation

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Example. (H-representation)

Let $P = \{x \mid A_P x \leq b_P\}$ and $Q = \{x \mid A_Q x \leq b_Q\}$ be polytopes. Then $x \in P \cap Q$?

Polytopes: H-representation and V-representation

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Solution: Linear programming! Check feasibility of

$$P \cap Q = \{x \mid A_P x \leq b_P, A_Q x \leq b_Q\}.$$

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Let $P = \text{conv}(U)$ and $Q = \text{conv}(W)$ be polytopes. Then $x \in P \cap Q$?

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$$\left\{(\lambda, \kappa) : \sum_{u \in U} \lambda_u u = \sum_{w \in W} \kappa_w w, \sum_{u \in U} \lambda_u = \sum_{w \in W} \kappa_w = 1, \lambda, \kappa \geq 0\right\}.$$

More general setup

Some trivial insights...

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Some trivial insights...

What if access to P and Q is only given implicitly?

What if P and Q are more general, e.g., compact convex?

von Neumann's Alternating Projections

The algorithm
von Neumann's Alternating Projections

Let P and Q be compact convex sets. Π_P, Π_Q being the respective projectors.

Algorithm von Neumann's Alternating Projections (POCS)

Input: Point $y_0 \in \mathbb{R}^n$, Π_P projector onto $P \subseteq \mathbb{R}^n$ and Π_Q projector onto $Q \subseteq \mathbb{R}^n$.
Output: Iterates $x_1, y_1 \ldots \in \mathbb{R}^n$

1: **for** $t = 0$ **to** $\ldots$ **do**
2: $\quad x_{t+1} \leftarrow \Pi_P(y_t)$
3: $\quad y_{t+1} \leftarrow \Pi_Q(x_{t+1})$

appeared in lecture notes first distributed in 1933; see reprint [von Neumann, 1949]

Convergence

von Neumann's Alternating Projections

Suppose $P \cap Q \neq \emptyset$ and let $u \in P \cap Q$. The binomial formula is your friend:

$$\|y_t - u\|^2$$

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$$\geq \|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2 + \|y_{t+1} - u\|^2.$$

Rearrange to

$$\|y_t - u\|^2 - \|y_{t+1} - u\|^2 \geq \|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2.$$

Convergence

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\begin{aligned}
\|y_t - u\|^2 = \|y_t - x_{t+1} + x_{t+1} - u\|^2 &= \|y_t - x_{t+1}\|^2 + \|x_{t+1} - u\|^2 - 2 \underbrace{\langle x_{t+1} - y_t, x_{t+1} - u \rangle}_{\leq 0} \\
&\geq \|y_t - x_{t+1}\|^2 + \|x_{t+1} - u\|^2 = \|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1} + y_{t+1} - u\|^2 \\
&= \|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2 + \|y_{t+1} - u\|^2 - 2 \underbrace{\langle y_{t+1} - x_{t+1}, y_{t+1} - u \rangle}_{\leq 0} \\
&\geq \|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2 + \|y_{t+1} - u\|^2.
\end{aligned}
$$

Rearrange to

$$\|y_t - u\|^2 - \|y_{t+1} - u\|^2 \geq \|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2.$$

Whenever you see something like this, it is checkmate in 3 moves...

Convergence

von Neumann's Alternating Projections

Starting from

$$\|y_t - u\|^2 - \|y_{t+1} - u\|^2 \geq \|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2.$$

1) Simply sum up

$$\sum_{t=0,\dots,T-1} \left(\|y_t - u\|^2 - \|y_{t+1} - u\|^2\right) \geq \sum_{t=0,\dots,T-1} \left(\|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2\right).$$

2) which implies, via telescoping,

$$\|y_0 - u\|^2 \geq \sum_{t=0,\dots,T-1} \left(\|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2\right).$$

3) divide by T, then

$$\frac{\|y_0 - u\|^2}{T} \geq \frac{1}{T} \sum_{t=0,\dots,T-1} \left(\|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2\right) \geq \|x_T - y_T\|^2,$$

as distances are non-increasing. □

Proposition (von Neumann [1949] + minor perturbations)

Let P and Q be compact convex sets with $P \cap Q \neq \emptyset$ and let $x_1, y_1 \ldots, x_T, y_T \in \mathbb{R}^n$ be the sequence of iterates of von Neumann's algorithm. Then the iterates converge: $x_t \longrightarrow x$ and $y_t \longrightarrow y$ to some $x \in P$ and $y \in Q$ and

$$\|x_T - y_T\|^2 \leq \frac{1}{T} \sum_{t=0}^{T-1} \left(\|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2 \right) \leq \frac{\operatorname{dist}(y_0, P \cap Q)^2}{T}.$$

Projections are often expensive however...

von Neumann's Alternating Projections

What if access to P and Q is only given by Linear Minimization Oracles (LMOs)?
(e.g., via combinatorial algorithm like matching algorithm)

Quick reminder. Linear minimization is often cheaper than projection (basically quadratic programming).

Alternating Linear Minimizations

[Braun et al., 2022]

von Neumann's algorithm revisited

Alternating Linear Minimizations

After close inspection and some meditation,

von Neumann's algorithm revisited

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After close inspection and some meditation, von Neumann's algorithm basically solves

$$\min_{(x,y)\in P\times Q} \|x - y\|^2,$$

i.e., we are minimizing the 2-norm over the product space $P \times Q$.

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In principle. Any Frank-Wolfe algorithm to solve the problem (only LMOs for P and Q).

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However. We want von Neumann style algorithm with alternations.

(Note. Above formulation might hint that acceleration is unlikely to be possible as condition number is 1.)

The Cyclic Block-Coordinate Conditional Gradient algorithm

Alternating Linear Minimizations

Luckily, [Beck et al., 2015] already thought about this...

Algorithm Cyclic Block-Coordinate Conditional Gradient algorithm [Beck et al., 2015]

Input: Points $x_i^0 \in P_i$, LMO for $P_i \subseteq \mathbb{R}^{n_i}$, $i = 0, \ldots, k-1$ and $0 < \gamma_0, \ldots, \gamma_t, \ldots \leq 1$.
Output: Iterates $x^1, \ldots \in P_0 \times \cdots \times P_{k-1}$

```
for t = 0 to ... do
    i ← t mod k
    v^t ← argmin_{x∈P_i} ⟨∇_{P_i} f(x^t), x⟩
    x^{t+1} ← x^t + γ_t (v^t − x_i^t)_[i]
```

1: **for** $t = 0$ **to** $\ldots$ **do**
2: $i \leftarrow t \bmod k$
3: $v^t \leftarrow \operatorname{argmin}_{x \in P_i} \langle \nabla_{P_i} f(x^t), x \rangle$
4: $x^{t+1} \leftarrow x^t + \gamma_t (v^t - x_i^t)_{[i]}$

Theorem (Convergence [Beck et al., 2015, cf Theorem 4.5])

Under standard assumptions

$$\text{(primal)} \quad f(x^{kt}) - f(x^*) \leq \frac{2}{t+2} \left(\sum_{i=0}^{k-1} \frac{L_i D_i^2}{2} + 2LD \sum_{i=0}^{k-1} D_i \right),$$

$$\text{(dual)} \quad \min_{1 \leq t \leq T} \max_{y \in P_0 \times \cdots \times P_{k-1}} \left\langle \nabla f(x^{kt}), x^{kt} - y \right\rangle \leq \frac{6.75}{T+2} \left(\sum_{i=0}^{k-1} \frac{L_i D_i^2}{2} + 2LD \sum_{i=0}^{k-1} D_i \right).$$

Note. Cyclic variant of stochastic BCFW [Lacoste-Julien et al., 2013]

Specializing Cyclic Block Coordinate Conditional Gradients [Beck et al., 2015]:

Alternating Linear Minimization algorithm

Alternating Linear Minimizations

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Algorithm Alternating Linear Minimizations (ALM)

Input: Points $x_0 \in P, y_0 \in Q$, LMO over $P, Q \subseteq \mathbb{R}^n$
Output: Iterates $x_1, y_1 \ldots \in \mathbb{R}^n$

1: **for** $t = 0$ **to** $\ldots$ **do**
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Observe.

1. Trivial algorithm: von Neumann + Sliding = inexact projection via FW requiring around $O(1/t)$ FW steps per iteration.
2. Here: Single(!!) Frank-Wolfe step on projection problem per iteration.

Convergence Guarantee

Alternating Linear Minimizations

Proposition (Intersection of two sets)

Let P and Q be compact convex sets. Then ALM generates iterates $z_t \doteq \frac{1}{2}(x_t + y_t)$, such that

$$\max\{\operatorname{dist}(z_t, P)^2, \operatorname{dist}(z_t, Q)^2\} \leq \frac{\|x_t - y_t\|^2}{4} \leq \frac{(1 + 2\sqrt{2})(D_P^2 + D_Q^2)}{t + 2} + \frac{\operatorname{dist}(P, Q)^2}{4}$$

$$\min_{1 \leq t \leq T} \max_{x \in P, y \in Q} \|x_t - y_t\|^2 - \langle x_t - y_t, x - y \rangle \leq \frac{6.75(1 + 2\sqrt{2})}{T + 2}(D_P^2 + D_Q^2).$$

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Note. *Rate is optimal, take $P = \Delta_n$ and $Q = \{0\} \Rightarrow$ standard lower bound for FW methods.*

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Remark (Comparison to von Neumann's alternating projection algorithm)

For simplicity let us consider the case where $P \cap Q \neq \emptyset$.
After minor reformulation, von Neumann's alternating projection method yields:

$$\min_{t=0,\ldots,T-1} \max\{\operatorname{dist}(z_{t+1}, P)^2, \operatorname{dist}(z_{t+1}, Q)^2\} \leq \frac{\operatorname{dist}(y_0, P \cap Q)^2}{T}.$$

Alternating Linear Minimization yields:

$$\max\{\operatorname{dist}(z_T, P)^2, \operatorname{dist}(z_T, Q)^2\} \leq \frac{(1 + 2\sqrt{2})(D_P^2 + D_Q^2)}{T + 2}.$$

All done?

Alternating Linear Minimizations

All done? We have been cheating however...

Alternating Linear Minimizations

Both von Neumann's algorithm and ALM only approximately decide $x \in P \cap Q$!

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Alternating Linear Minimizations

Both von Neumann's algorithm and ALM only approximately decide $x \in P \cap Q$!

For general compact convex sets this is as good as it gets but for polytopes?

Alternating Linear Minimizations for Polytopes

[Braun et al., 2022]

A simply observation

Alternating Linear Minimizations for Polytopes

Observation (Approximate-Exact Crossover)

Let $P, Q \subseteq \mathbb{R}^n$ be polytopes. There exists $\varepsilon_{PQ} > 0$, so that for all $U \subseteq \mathrm{vert}(P)$, $V \subseteq \mathrm{vert}(Q)$ with $\mathrm{dist}(\mathrm{conv}(U), \mathrm{conv}(V)) < \varepsilon_{PQ}$, it holds $\mathrm{conv}(U) \cap \mathrm{conv}(V) \neq \emptyset$.

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Proof.

Follows from the fact that polytopes having only a finite number of vertices:

$$\varepsilon_{PQ} := \min\{\mathrm{dist}(\mathrm{conv}(U), \mathrm{conv}(V)) : U \subseteq \mathrm{vert}(P), V \subseteq \mathrm{vert}(Q), \mathrm{conv}(U) \cap \mathrm{conv}(V) = \emptyset\}.$$

□

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□

Of course we do not know ε_{PQ} ahead of time...

Another simple observation

Alternating Linear Minimizations for Polytopes

Observation (Recovery of $x \in P \cap Q$ by linear programming)

Assume x_t and y_t with $\|x_t - y_t\| < \varepsilon_{PQ}$ via ALM.

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Assume x_t and y_t with $\|x_t - y_t\| < \varepsilon_{PQ}$ via ALM.

Let $U \subseteq \mathrm{vert}(P)$ be all extreme points returned by the LMO for P throughout the execution of ALM and define $V \subseteq \mathrm{vert}(Q)$ accordingly. From Observation: $\mathrm{conv}(U) \cap \mathrm{conv}(V) \neq \emptyset$.

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Solve linear feasibility program

$$\sum_{u \in U} \lambda_u u = \sum_{v \in V} \kappa_u v$$
$$\sum_{u \in U} \lambda_u = 1, \sum_{v \in V} \kappa_u = 1$$
$$\lambda \geq 0, \kappa \geq 0,$$

to obtain

$$x := \sum_{u \in U} \lambda_u u = \sum_{v \in V} \kappa_u v \in P \cap Q.$$

An exact algorithm

Alternating Linear Minimizations for Polytopes

Algorithm Alternating Linear Minimizations (ALM) [exact version]

Input: Points $x_0 \in P$, $y_0 \in Q$, LMO over $P, Q \subseteq \mathbb{R}^n$
Output: Iterates $x_1, y_1 \ldots \in \mathbb{R}^n$

for $t = 0$ **to** $\ldots$ **do**
    $u_t \leftarrow \operatorname{argmin}_{x \in P} \langle x_t - y_t, x \rangle$
    $x_{t+1} \leftarrow x_t + \frac{2}{t+2} \cdot (u_t - x_t)$
    $v_t \leftarrow \operatorname{argmin}_{y \in Q} \langle y_t - x_{t+1}, y \rangle$
    $y_{t+1} \leftarrow y_t + \frac{2}{t+2} \cdot (v_t - y_t)$
    if $t = 2^k$ for some k **then**
        if $\min_{x \in P, y \in Q} \langle x_{t+1} - y_{t+1}, x - y \rangle > 0$ **then**
            return “disjoint” and certificate $\langle x_{t+1} - y_{t+1}, x - y \rangle > 0$
        else
            Solve linear feasibility program.
            if feasible **then**
                return a solution $x \in P \cap Q$

An exact algorithm: Guarantees

Alternating Linear Minimizations for Polytopes

Basically we pay a factor of 2 in iterations for making exact.

Proposition (Exact variant)

Let P, Q be polytopes with diameters D_p and D_Q, respectively. Executing exact ALM variant:

1. *If $P \cap Q \neq \emptyset$, then after no more than*

$$\frac{16(1+2\sqrt{2})(D_P^2 + D_Q^2)}{\varepsilon_{PQ}^2}$$

   block-LMO calls, the algorithm returns $x \in P \cap Q$.

2. *If $P \cap Q = \emptyset$, then after no more than*

$$16(1+2\sqrt{2})(D_P^2 + D_Q^2)\frac{(D_P + D_Q)^2}{\operatorname{dist}(P, Q)^4}$$

   block-LMO calls the algorithm certifies $P \cap Q = \emptyset$.

Note. We counted the resolution of one feasibility LP as one block-LMO.

How bad can it be?

Bounds on minimal distance

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The minimal possible distance $\mathrm{dist}(P, Q)$ can be actually quite bad.

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Theorem

If P and Q are disjoint lattice (d, k)-polytopes, then

$$\frac{1}{(kd)^{2d}} \leq \operatorname{dist}(P, Q),$$

and for any large enough d, there exist two disjoint (d, k)-lattice polytopes P and Q such that

$$\operatorname{dist}(P, Q) \leq \frac{1}{\left(k\sqrt{d}\right)^{\sqrt{d}}}.$$

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⇒ In case of disjoint polytopes running time can be as bad as

$$\Omega\left(\left(k\sqrt{d}\right)^{4\sqrt{d}}\right).$$

⇒ Bad news for our algorithms.

Can we do better?

Advanced FW algorithms over polytopes
Can we do better?

AFW, PCG, BCG, BPCG, etc. can solve

$$\min_{x \in P} f(x),$$

to accuracy ε in roughly

$$\mathcal{O}\left(\frac{LD^2}{\mu\delta^2} \log \frac{1}{\varepsilon}\right)$$

iterations, for f being L-smooth and μ-PL over a polytope P with pyramidal width δ.

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Recall. Our problem can be formulated as

$$\min_{(x,y) \in P \times Q} \|x - y\|^2,$$

which is 1-smooth and 1-PL (basically "like" strong-convexity).

Pyramidal width Over Products

Can we do better?

With a little bit of geometric reasoning we can show: [Iommazzo et al., 2025]

Theorem (Pyramidal width of the product)

Let δ_P and δ_Q be the pyramidal widths of polytopes $P, Q \subseteq \mathbb{R}^n$. Then,

$$\delta_{P\times Q} = \sqrt{\frac{\delta_P^2 \delta_Q^2}{\delta_P^2 + \delta_Q^2}}.$$

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Corollary (Useful lower bound for the pyramidal width of the product)

The pyramidal width of product polytope $P = \prod_{i \in [k]} P_i$ is at least

$$\delta_P = \Omega\left\{\frac{1}{\sqrt{k}} \min_{i \in [k]} \delta_{P_i}\right\}$$

with $\delta_{\mathcal{P}_i}$ being pyramidal width of P_i; bound is essentially tight when one pyramidal width is much smaller than the others.

Putting it all together

Can we do better?

[Iommazzo et al., 2025]

Proposition (Faster exact variant)

Let P, Q be polytopes with diameters D_p and D_Q, respectively. Executing exact ALM variant with AFW, PCG, BCG, BPCG, etc. steps:

1. *If $P \cap Q \neq \emptyset$, then the algorithm returns $x \in P \cap Q$ in*

$$\mathcal{O}\left(\frac{D_P^2 D_Q^2}{\min\{\delta_P, \delta_Q\}^2} \log \frac{1}{\varepsilon_{PQ}}\right).$$

2. *If $P \cap Q = \emptyset$, then the algorithm certifies $P \cap Q = \emptyset$ in*

$$\mathcal{O}\left(\frac{D_P^2 D_Q^2}{\min\{\delta_P, \delta_Q\}^2} \log \frac{1}{\mathrm{dist}(P, Q)}\right).$$

Note. $D_P, D_Q, \delta_P, \delta_Q$ are translation invariant and only depend on P and Q, respectively.

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Note. $D_P, D_Q, \delta_P, \delta_Q$ are translation invariant and only depend on P and Q, respectively.
Worst-case example from before. Running time reduces to

$$\mathcal{O}\left(\frac{D_P^2 D_Q^2}{\min\{\delta_P, \delta_Q\}^2} \sqrt{d} \log k\sqrt{d}\right).$$

Outlook

Integrating LP solving into convex optimization is very powerful

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[Halbey et al., 2025]

In Entanglement Detection, Sliding, Splitting, etc. we encounter quadratic programs.

⇒ First-order optimality system is a linear program!

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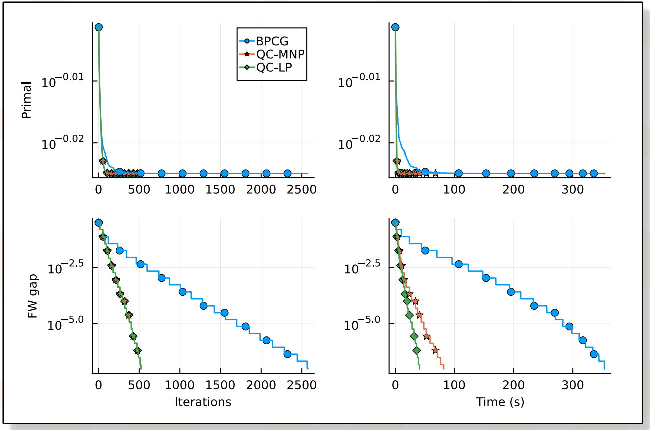
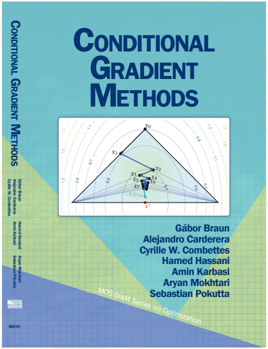


Integration of LP solving into convex optimization is very powerful.

If you want to learn more...

Thank you!



Conditional Gradient Methods

Gábor Braun, Alejandro Carderera, Cyrille W Combettes, Hamed Hassani, Amin Karbasi, Aryan Mokhtari, and Sebastian Pokutta

`https://conditional-gradients.org/`
`https://arxiv.org/abs/2211.14103`

to appear in MOS-SIAM Series on Optimization

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