Alternating Linear Minimization: Revisiting von Neumann's alternating projections

Sebastian Pokutta

joint work with: Gábor Braun and Robert Weismantel

Technische Universität Berlin
and
Zuse Institute Berlin

pokutta@math.tu-berlin.de
@spokutta

Halfway to Twelfty

July 4th, 2023 · Paris, France

Berlin Mathematics Research Center

What is this talk about?

Introduction

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does there exist $x \in P \cap Q$?

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Why? At the core of many algorithms. Allows for optimization via binary search.

Today: von Neumann's approach and a new algorithm.

(Hyperlinked) References are not exhaustive; check references contained therein.

Some trivial insights...

Polytopes: H-representation and V-representation

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Example. (H-representation)

Let $P = \{x \mid A_P x \leq b_P\}$ and $Q = \{x \mid A_Q x \leq b_Q\}$ be polytopes. Then $x \in P \cap Q$?

Polytopes: H-representation and V-representation

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$$P \cap Q = \{x \mid A_P x \leq b_P, A_Q x \leq b_Q\}.$$

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$$\left\{(\lambda, \kappa) : \sum_{u \in U} \lambda_u u = \sum_{w \in W} \kappa_w w, \sum_{u \in U} \lambda_u = \sum_{w \in W} \kappa_w = 1, \lambda, \kappa \geq 0\right\}.$$

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What if access to P and Q is only given implicitly?

What if P and Q are more general, e.g., compact convex?

von Neumann's Alternating Projections

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Algorithm von Neumann's Alternating Projections (POCS)

Input: Point $y_0 \in \mathbb{R}^n$, Π_P projector onto $P \subseteq \mathbb{R}^n$ and Π_Q projector onto $Q \subseteq \mathbb{R}^n$.
Output: Iterates $x_1, y_1 \ldots \in \mathbb{R}^n$

1: **for** $t = 0$ **to** $\ldots$ **do**
2: $\quad x_{t+1} \leftarrow \Pi_P(y_t)$
3: $\quad y_{t+1} \leftarrow \Pi_Q(x_{t+1})$

appeared in lecture notes first distributed in 1933; see reprint [von Neumann, 1949]

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$$\|y_t - u\|^2$$

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&\geq \|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2 + \|y_{t+1} - u\|^2.
\end{aligned}$$

Rearrange to

$$\|y_t - u\|^2 - \|y_{t+1} - u\|^2 \geq \|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2.$$

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Whenever you see something like this, it is checkmate in 3 moves...

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1) Simply sum up

$$\sum_{t=0,\ldots,T-1} \left(\|y_t - u\|^2 - \|y_{t+1} - u\|^2\right) \geq \sum_{t=0,\ldots,T-1} \left(\|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2\right).$$

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2) which implies, via telescoping,

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3) divide by T, then

$$\frac{\|y_0 - u\|^2}{T} \geq \frac{1}{T} \sum_{t=0,\dots,T-1} \left(\|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2\right)$$

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as distances are non-increasing. □

Convergence

von Neumann's Alternating Projections

Proposition (von Neumann [1949] + minor perturbations)

Let P and Q be compact convex sets with $P \cap Q \neq \emptyset$ and let $x_1, y_1 \ldots, x_T, y_T \in \mathbb{R}^n$ be the sequence of iterates of von Neumann's algorithm. Then the iterates converge: $x_t \longrightarrow x$ and $y_t \longrightarrow y$ to some $x \in P$ and $y \in Q$ and

$$\|x_T - y_T\|^2 \leq \frac{1}{T} \sum_{t=0}^{T-1} \left(\|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2 \right) \leq \frac{\operatorname{dist}(y_0, P \cap Q)^2}{T}.$$

Projections are often expensive however...

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What if access to P and Q is only given by Linear Minimization Oracles (LMOs)?
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Quick reminder. Linear minimization is often cheaper than projection (basically quadratic programming).

Alternating Linear Minimizations

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von Neumann's algorithm revisited

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After close inspection and some meditation, von Neumann's algorithm basically solves

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[Braun et al., 2022]

von Neumann's algorithm revisited

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[Braun et al., 2022]

However. We want von Neumann style algorithm with alternations.

(Note. Above formulation might hint that acceleration is unlikely to be possible as condition number is 1.)

The Cyclic Block-Coordinate Conditional Gradient algorithm

Luckily, [Beck et al., 2015] already thought about this...

Algorithm Cyclic Block-Coordinate Conditional Gradient algorithm [Beck et al., 2015]

Input: Points $x_i^0 \in P_i$, LMO for $P_i \subseteq \mathbb{R}^{n_i}$, $i = 0, \dots, k-1$ and $0 < \gamma_0, \dots, \gamma_t, \dots \leq 1$.
Output: Iterates $x^1, \dots \in P_0 \times \cdots \times P_{k-1}$

---

for $t = 0$ **to** $\dots$ **do**
    $i \leftarrow t \bmod k$
    $v^t \leftarrow \operatorname{argmin}_{x \in P_i} \left\langle \nabla_{P_i} f(x^t), x \right\rangle$
    $x^{t+1} \leftarrow x^t + \gamma_t (v^t - x_i^t)_{[i]}$

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Theorem (Convergence [Beck et al., 2015, cf Theorem 4.5])

Under standard assumptions

$$\textit{(primal)} \quad f(x^{kt}) - f(x^*) \leq \frac{2}{t+2} \left(\sum_{i=0}^{k-1} \frac{L_i D_i^2}{2} + 2LD \sum_{i=0}^{k-1} D_i \right),$$

$$\textit{(dual)} \quad \min_{1 \leq t \leq T} \max_{y \in P_0 \times \cdots \times P_{k-1}} \left\langle \nabla f(x^{kt}), x^{kt} - y \right\rangle \leq \frac{6.75}{T+2} \left(\sum_{i=0}^{k-1} \frac{L_i D_i^2}{2} + 2LD \sum_{i=0}^{k-1} D_i \right).$$

Note. Cyclic variant of stochastic BCFW [Lacoste-Julien et al., 2013]

Specializing Cyclic Block Coordinate Conditional Gradients [Beck et al., 2015]:

Alternating Linear Minimization algorithm

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Algorithm Alternating Linear Minimizations (ALM)

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Output: Iterates $x_1, y_1 \ldots \in \mathbb{R}^n$

for $t = 0$ **to** … **do**
  $u_t \leftarrow \text{argmin}_{x \in P} \langle x_t - y_t, x \rangle$
  $x_{t+1} \leftarrow x_t + \frac{2}{t+2} \cdot (u_t - x_t)$
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Observe.

1. Trivial algorithm: von Neumann + Sliding = inexact projection via FW requiring around $O(1/t)$ FW step per iteration.
2. Here: Single(!!) Frank-Wolfe step on projection problem per iteration.

Convergence Guarantee

Alternating Linear Minimizations

Proposition (Intersection of two sets)

Let P and Q be compact convex sets. Then ALM generates iterates $z_t \doteq \frac{1}{2}(x_t + y_t)$*, such that*

$$\max\{\operatorname{dist}(z_t, P)^2, \operatorname{dist}(z_t, Q)^2\} \leq \frac{\|x_t - y_t\|^2}{4} \leq \frac{(1 + 2\sqrt{2})(D_P^2 + D_Q^2)}{t+2} + \frac{\operatorname{dist}(P, Q)^2}{4}$$

$$\min_{1 \leq t \leq T} \max_{x \in P, y \in Q} \|x_t - y_t\|^2 - \langle x_t - y_t, x - y \rangle \leq \frac{6.75(1 + 2\sqrt{2})}{T+2}(D_P^2 + D_Q^2).$$

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Note. *Rate is optimal, take $P = \Delta_n$ and $Q = \{0\} \Rightarrow$ standard lower bound for FW methods.*

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Remark (Comparison to von Neumann's alternating projection algorithm)

For simplicity let us consider the case where $P \cap Q \neq \emptyset$.
After minor reformulation, von Neumann's alternating projection method yields:

$$\min_{t=0,\ldots,T-1} \max\{\operatorname{dist}(z_{t+1}, P)^2, \operatorname{dist}(z_{t+1}, Q)^2\} \leq \frac{\operatorname{dist}(y_0, P \cap Q)^2}{T}.$$

Alternating Linear Minimization yields:

$$\max\{\operatorname{dist}(z_T, P)^2, \operatorname{dist}(z_T, Q)^2\} \leq \frac{(1 + 2\sqrt{2})(D_P^2 + D_Q^2)}{T+2}.$$

Convergence for $P \cap Q = \emptyset$

Alternating Linear Minimizations

Corollary (Certifying that $P \cap Q = \emptyset$)

If some iterates of ALM satisfy

$$\|x_t - y_t\|^2 > \frac{4(1 + 2\sqrt{2})(D_P^2 + D_Q^2)}{t + 2}$$

then $P \cap Q = \emptyset$. If $P \cap Q = \emptyset$ then the above condition is satisfied after at most

$$\frac{8(1 + 2\sqrt{2})(D_P^2 + D_Q^2)}{\mathrm{dist}(P, Q)^2}$$

block-LMO calls.

Convergence for $P \cap Q = \emptyset$ without knowledge of D_P and D_Q

Corollary (Certifying $P \cap Q = \emptyset$ without knowledge of D_P and D_Q)

Then executing ALM, after at most

$$\frac{13.5(1+2\sqrt{2})(D_P^2 + D_Q^2)}{\operatorname{dist}(P,Q)^2}$$

block-LMO calls, some (of the already seen!) iteration t provides the following certificate for disjointness, which does not require explicit bounds on D_P and D_Q:

$$\min_{x \in P, y \in Q} \langle x_t - y_t, x - y \rangle > 0.$$

Moreover, this inequality is guaranteed to hold for every iteration

$$t > 4(1+2\sqrt{2})(D_P^2 + D_Q^2)(D_P + D_Q)^2 / \operatorname{dist}(P,Q)^4.$$

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$$\frac{13.5(1 + 2\sqrt{2})(D_P^2 + D_Q^2)}{\operatorname{dist}(P, Q)^2}$$

block-LMO calls, some (of the already seen!) iteration t provides the following certificate for disjointness, which does not require explicit bounds on D_P and D_Q:

$$\min_{x \in P, y \in Q} \langle x_t - y_t, x - y \rangle > 0.$$

Moreover, this inequality is guaranteed to hold for every iteration

$$t > 4(1 + 2\sqrt{2})(D_P^2 + D_Q^2)(D_P + D_Q)^2 / \operatorname{dist}(P, Q)^4.$$

Note. Testing in each iteration would be inefficient (additional LMO call).

All done?

Alternating Linear Minimizations

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Both von Neumann's algorithm and ALM only approximately decide $x \in P \cap Q$!

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For general compact convex sets this is as good as it gets but for polytopes?

Alternating Linear Minimizations for Polytopes

A simply observation

Alternating Linear Minimizations for Polytopes

Observation (Approximate-Exact Crossover)

Let $P, Q \subseteq \mathbb{R}^n$ *be polytopes. There exists* $\varepsilon_{PQ} > 0$*, so that for all* $U \subseteq \mathrm{vert}(P), V \subseteq \mathrm{vert}(Q)$ *with* $\mathrm{dist}(\mathrm{conv}(U), \mathrm{conv}(V)) < \varepsilon_{PQ}$*, it holds* $\mathrm{conv}(U) \cap \mathrm{conv}(V) \neq \emptyset$*.*

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Proof.

Follows from the fact that polytopes having only a finite number of vertices:

$$\varepsilon_{PQ} := \min\{\operatorname{dist}(\operatorname{conv}(U), \operatorname{conv}(V)) : U \subseteq \operatorname{vert}(P), V \subseteq \operatorname{vert}(Q), \operatorname{conv}(U) \cap \operatorname{conv}(V) = \emptyset\}.$$

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□

Of course we do not know ε_{PQ} ahead of time...

Another simple observation

Alternating Linear Minimizations for Polytopes

Observation (Recovery of $x \in P \cap Q$ by linear programming)

Assume x_t and y_t with $\|x_t - y_t\| < \varepsilon_{PQ}$ via ALM.

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Solve linear feasibility program

$$\sum_{u \in U} \lambda_u u = \sum_{v \in V} \kappa_u v$$

$$\sum_{u \in U} \lambda_u = 1, \sum_{v \in V} \kappa_u = 1$$

$$\lambda \geq 0, \kappa \geq 0,$$

to obtain

$$x := \sum_{u \in U} \lambda_u u = \sum_{v \in V} \kappa_u v \in P \cap Q.$$

An exact algorithm

Alternating Linear Minimizations for Polytopes

Algorithm Alternating Linear Minimizations (ALM) [exact version]

Input: Points $x_0 \in P, y_0 \in Q$, LMO over $P, Q \subseteq \mathbb{R}^n$
Output: Iterates $x_1, y_1 \ldots \in \mathbb{R}^n$

for $t = 0$ **to** ... **do**
  $u_t \leftarrow \operatorname{argmin}_{x \in P} \langle x_t - y_t, x \rangle$
  $x_{t+1} \leftarrow x_t + \frac{2}{t+2} \cdot (u_t - x_t)$
  $v_t \leftarrow \operatorname{argmin}_{y \in Q} \langle y_t - x_{t+1}, y \rangle$
  $y_{t+1} \leftarrow y_t + \frac{2}{t+2} \cdot (v_t - y_t)$
  if $t = 2^k$ for some k **then**
    if $\min_{x \in P, y \in Q} \langle x_{t+1} - y_{t+1}, x - y \rangle > 0$ **then**
      return "disjoint" and certificate $\langle x_{t+1} - y_{t+1}, x - y \rangle > 0$
    else
      Solve linear feasibility program.
      if feasible **then**
        return a solution $x \in P \cap Q$

An exact algorithm: Guarantees

Alternating Linear Minimizations for Polytopes

Basically we pay a factor of 2 in iterations for making exact.

Proposition (Exact variant)

Let P, Q be polytopes with diameters D_p and D_Q, respectively. Executing exact ALM variant:

1. *If $P \cap Q \neq \emptyset$, then after no more than*

$$\frac{16(1+2\sqrt{2})(D_P^2 + D_Q^2)}{\varepsilon_{PQ}^2}$$

   block-LMO calls, the algorithm returns $x \in P \cap Q$.

2. *If $P \cap Q = \emptyset$, then after no more than*

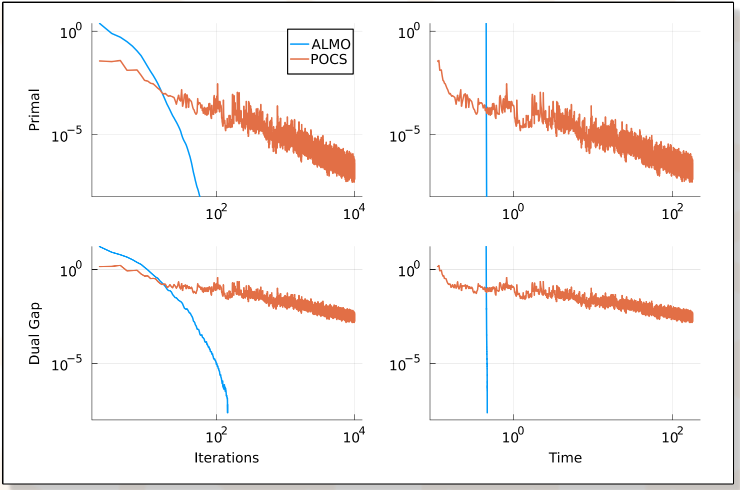
$$16(1+2\sqrt{2})(D_P^2 + D_Q^2)\frac{(D_P + D_Q)^2}{\operatorname{dist}(P,Q)^4}$$

   block-LMO calls the algorithm certifies $P \cap Q = \emptyset$.

Note. We counted the resolution of one feasibility LP as one block-LMO.

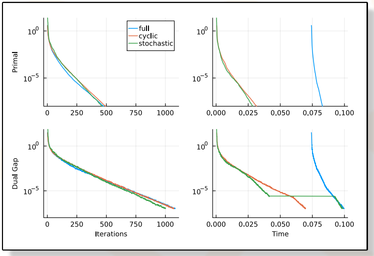
Some ~~cooked~~ preliminary computational results...

ALMO vs. POCS

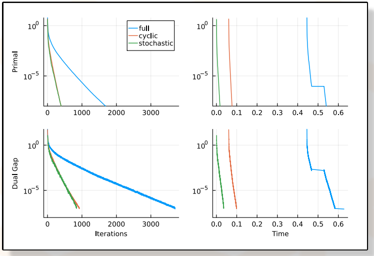


Intersection of two polytopes. Projection problem solved approximately via FW; relevant for time, irrelevant for iterations.

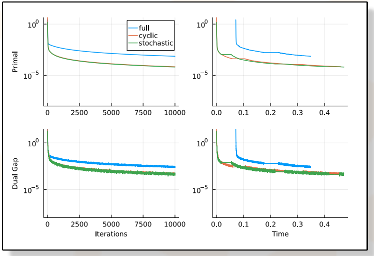
Full vs. stochastic vs. cyclic



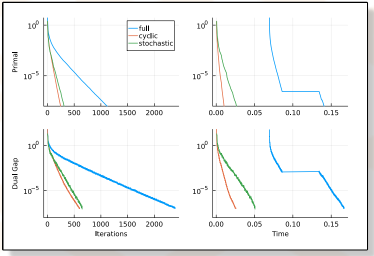
L_1-ball and random polytope (intersecting)



two random polytopes (non-intersecting)

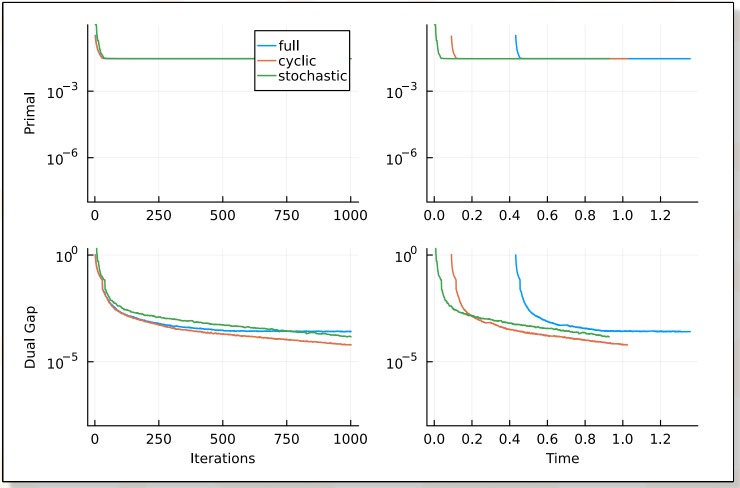


L_1-ball and random polytope (intersecting)



three random polytopes

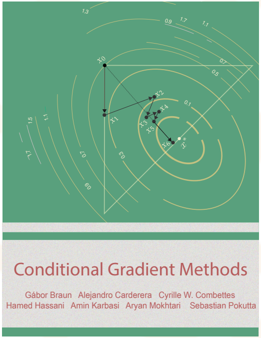
Solving SDPs



Intersection of SDP cone with polytope

Shameless plug...

Thank you!



Conditional Gradient Methods

Gábor Braun, Alejandro Carderera, Cyrille W Combettes, Hamed Hassani, Amin Karbasi, Aryan Mokhtari, and Sebastian Pokutta

`https://conditional-gradients.org/`
`https://arxiv.org/abs/2211.14103`

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