Alternating Linear Minimization:
Revisiting von Neumann's alternating projections

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What is this talk about?

Introduction

Given P, Q compact convex sets,
does there exist $x \in P \cap Q$?

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Why? At the core of many algorithms. Allows for optimization via binary search.

Today: von Neumann's approach and a new algorithm.

(Hyperlinked) References are not exhaustive; check references contained therein.

Some trivial insights...

Polytopes: H-representation and V-representation

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Example. (H-representation)

Let $P = \{x \mid A_P x \leq b_P\}$ and $Q = \{x \mid A_Q x \leq b_Q\}$ be polytopes. Then $x \in P \cap Q$?

Polytopes: H-representation and V-representation

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Let $P = \text{conv}(U)$ and $Q = \text{conv}(W)$ be polytopes. Then $x \in P \cap Q$?

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$$\left\{(\lambda, \kappa) : \sum_{u \in U} \lambda_u u = \sum_{w \in W} \kappa_w w, \sum_{u \in U} \lambda_u = \sum_{w \in W} \kappa_w = 1, \lambda, \kappa \geq 0\right\}.$$

More general setup

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What if access to P and Q is only given implicitly?

What if P and Q are more general, e.g., compact convex?

von Neumann's Alternating Projections

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Algorithm von Neumann's Alternating Projections

Input: Point $y_0 \in \mathbb{R}^n$, Π_P projector onto $P \subseteq \mathbb{R}^n$ and Π_Q projector onto $Q \subseteq \mathbb{R}^n$.
Output: Iterates $x_1, y_1 \ldots \in \mathbb{R}^n$

for $t = 0$ **to** ... **do**
    $x_{t+1} \leftarrow \Pi_P(y_t)$
    $y_{t+1} \leftarrow \Pi_Q(x_{t+1})$

appeared in lecture notes first distributed in 1933; see reprint [von Neumann, 1949]

Convergence

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&\geq \|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2 + \|y_{t+1} - u\|^2.
\end{aligned}$$

Rearrange to

$$\|y_t - u\|^2 - \|y_{t+1} - u\|^2 \geq \|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2.$$

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Whenever you see something like this, it is checkmate in 3 moves...

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1) Simply sum up

$$\sum_{t=0,\ldots,T-1} \left(\|y_t - u\|^2 - \|y_{t+1} - u\|^2\right) \geq \sum_{t=0,\ldots,T-1} \left(\|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2\right).$$

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2) which implies, via telescoping,

$$\|y_0 - u\|^2 \geq \sum_{t=0,\dots,T-1} \left(\|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2\right).$$

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3) divide by T, then

$$\frac{\|y_0 - u\|^2}{T} \geq \frac{1}{T} \sum_{t=0,\ldots,T-1} \left(\|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2\right)$$

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as distances are non-increasing. □

Convergence

von Neumann's Alternating Projections

Proposition (von Neumann [1949] + minor perturbations)

Let P and Q be compact convex sets with $P \cap Q \neq \emptyset$ and let $x_1, y_1 \ldots, x_T, y_T \in \mathbb{R}^n$ be the sequence of iterates of von Neumann's algorithm. Then the iterates converge: $x_t \longrightarrow x$ and $y_t \longrightarrow y$ to some $x \in P$ and $y \in Q$ and

$$\|x_T - y_T\|^2 \leq \frac{1}{T} \sum_{t=0}^{T-1} \left(\|y_t - x_{t+1}\|^2 + \|x_{t+1} - y_{t+1}\|^2 \right) \leq \frac{\operatorname{dist}(y_0, P \cap Q)^2}{T}.$$

Projections are often expensive however...

von Neumann's Alternating Projections

What if access to P and Q is only given by Linear Minimization Oracles (LMOs)?

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Quick reminder. Linear minimization is often cheaper than projection (basically quadratic programming).

Alternating Linear Minimizations

von Neumann's algorithm revisited

Alternating Linear Minimizations

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[Braun et al., 2022]

von Neumann's algorithm revisited

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[Braun et al., 2022]

However. We want von Neumann style algorithm with alternations.

The Cyclic Block-Coordinate Conditional Gradient algorithm

Alternating Linear Minimizations

Algorithm Cyclic Block-Coordinate Conditional Gradient algorithm [Beck et al., 2015]

Input: Points $x_i^0 \in P_i$, LMO for $P_i \subseteq \mathbb{R}^{n_i}$, $i = 0, \ldots, k-1$ and $0 < \gamma_0, \ldots, \gamma_t, \ldots \leq 1$.

Output: Iterates $x^1, \ldots \in P_0 \times \cdots \times P_{k-1}$

for $t = 0$ **to** $\ldots$ **do**

$\quad i \leftarrow t \bmod k$

$\quad v^t \leftarrow \operatorname{argmin}_{x \in P_i} \left\langle \nabla_{P_i} f(x^t), x \right\rangle$

$\quad x^{t+1} \leftarrow x^t + \gamma_t (v^t - x_i^t)_{[i]}$

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4: $\quad x^{t+1} \leftarrow x^t + \gamma_t (v^t - x_i^t)_{[i]}$

Theorem (Convergence [Beck et al., 2015, cf Theorem 4.5])

Under standard assumptions

$$\text{(primal)} \quad f(x^{kt}) - f(x^*) \leq \frac{2}{t+2} \left(\sum_{i=0}^{k-1} \frac{L_i D_i^2}{2} + 2LD \sum_{i=0}^{k-1} D_i \right),$$

$$\text{(dual)} \quad \min_{1 \leq t \leq T} \max_{y \in P_0 \times \cdots \times P_{k-1}} \left\langle \nabla f(x^{kt}), x^{kt} - y \right\rangle \leq \frac{6.75}{T+2} \left(\sum_{i=0}^{k-1} \frac{L_i D_i^2}{2} + 2LD \sum_{i=0}^{k-1} D_i \right).$$

Specializing the Cyclic Block Coordinate Conditional Gradient algorithm:

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Algorithm Alternating Linear Minimizations (ALM)

Input: Points $x_0 \in P$, $y_0 \in Q$, LMO over $P, Q \subseteq \mathbb{R}^n$
Output: Iterates $x_1, y_1 \ldots \in \mathbb{R}^n$

1: **for** $t = 0$ **to** ... **do**
2: $u_t \leftarrow \operatorname{argmin}_{x \in P} \langle x_t - y_t, x \rangle$
3: $x_{t+1} \leftarrow x_t + \frac{2}{t+2} \cdot (u_t - x_t)$
4: $v_t \leftarrow \operatorname{argmin}_{y \in Q} \langle y_t - x_{t+1}, y \rangle$
5: $y_{t+1} \leftarrow y_t + \frac{2}{t+2} \cdot (v_t - y_t)$

Convergence for $P \cap Q \neq \emptyset$

Proposition (Intersection of two sets)

Let P and Q be compact convex sets. Then ALM generates iterates $z_t \doteq \frac{1}{2}(x_t + y_t)$, *such that*

$$\max\{\operatorname{dist}(z_t, P)^2, \operatorname{dist}(z_t, Q)^2\} \leq \frac{\|x_t - y_t\|^2}{4} \leq \frac{(1+2\sqrt{2})(D_P^2 + D_Q^2)}{t+2} + \frac{\operatorname{dist}(P,Q)^2}{4}$$

$$\min_{1 \leq t \leq T} \max_{x \in P, y \in Q} \|x_t - y_t\|^2 - \langle x_t - y_t, x - y \rangle \leq \frac{6.75(1+2\sqrt{2})}{T+2}(D_P^2 + D_Q^2).$$

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Remark (Comparison to von Neumann's alternating projection algorithm)

For simplicity let us consider the case where $P \cap Q \neq \emptyset$.
After minor reformulation, von Neumann's alternating projection method yields:

$$\min_{t=0,\ldots,T-1} \max\{\mathrm{dist}(z_{t+1}, P)^2, \mathrm{dist}(z_{t+1}, Q)^2\} \leq \frac{\mathrm{dist}(y_0, P \cap Q)^2}{T}.$$

Alternating Linear Minimization yields:

$$\max\{\mathrm{dist}(z_T, P)^2, \mathrm{dist}(z_T, Q)^2\} \leq \frac{(1+2\sqrt{2})(D_P^2 + D_Q^2)}{T+2}.$$

Convergence for $P \cap Q = \emptyset$

Alternating Linear Minimizations

Corollary (Certifying that $P \cap Q = \emptyset$)

If some iterates of ALM satisfy

$$\|x_t - y_t\|^2 > \frac{4(1 + 2\sqrt{2})(D_P^2 + D_Q^2)}{t+2}$$

then $P \cap Q = \emptyset$. If $P \cap Q = \emptyset$ then the above condition is satisfied after at most

$$\frac{8(1 + 2\sqrt{2})(D_P^2 + D_Q^2)}{\text{dist}(P, Q)^2}$$

block-LMO calls.

Corollary (Certifying $P \cap Q = \emptyset$ without knowledge of D_P and D_Q)

Then executing ALM, after at most

$$\frac{13.5(1 + 2\sqrt{2})(D_P^2 + D_Q^2)}{\operatorname{dist}(P, Q)^2}$$

block-LMO calls, some (of the already seen!) iteration t provides the following certificate for disjointness, which does not require explicit bounds on D_P and D_Q:

$$\min_{x \in P, y \in Q} \langle x_t - y_t, x - y \rangle > 0.$$

Moreover, this inequality is guaranteed to hold for every iteration

$$t > 4(1 + 2\sqrt{2})(D_P^2 + D_Q^2)(D_P + D_Q)^2 / \operatorname{dist}(P, Q)^4.$$

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Note. Testing in each iteration would be inefficient (additional LMO call).

All done?

Alternating Linear Minimizations

All done? We have been cheating however...

Both von Neumann's algorithm and ALM only approximately decide $x \in P \cap Q$!

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Alternating Linear Minimizations

Both von Neumann's algorithm and ALM only approximately decide $x \in P \cap Q$!

For general compact convex sets this is as good as it gets but for polytopes?

Alternating Linear Minimizations for Polytopes

A simply observation

Alternating Linear Minimizations for Polytopes

Observation (Approximate-Exact Crossover)

Let $P, Q \subseteq \mathbb{R}^n$ be polytopes. There exists $\varepsilon_{PQ} > 0$, so that for all $U \subseteq \operatorname{vert}(P)$, $V \subseteq \operatorname{vert}(Q)$ with $\operatorname{dist}(\operatorname{conv}(U), \operatorname{conv}(V)) < \varepsilon_{PQ}$, it holds $\operatorname{conv}(U) \cap \operatorname{conv}(V) \neq \emptyset$.

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Proof.

Follows from the fact that polytopes having only a finite number of vertices:

$$\varepsilon_{PQ} \coloneqq \min\{\operatorname{dist}(\operatorname{conv}(U), \operatorname{conv}(V)) : U \subseteq \operatorname{vert}(P), V \subseteq \operatorname{vert}(Q), \operatorname{conv}(U) \cap \operatorname{conv}(V) = \emptyset\}.$$

□

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Proof.

Follows from the fact that polytopes having only a finite number of vertices:

$$\varepsilon_{PQ} \coloneqq \min\{\mathrm{dist}(\mathrm{conv}(U), \mathrm{conv}(V)) : U \subseteq \mathrm{vert}(P), V \subseteq \mathrm{vert}(Q), \mathrm{conv}(U) \cap \mathrm{conv}(V) = \emptyset\}.$$

□

Of course we do not know ε_{PQ} ahead of time...

Another simple observation

Alternating Linear Minimizations for Polytopes

Observation (Recovery of $x \in P \cap Q$ by linear programming)

Assume x_t and y_t with $\|x_t - y_t\| < \varepsilon_{PQ}$ via ALM.

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Let $U \subseteq \mathrm{vert}(P)$ be all extreme points returned by the LMO for P throughout the execution of ALM and define $V \subseteq \mathrm{vert}(Q)$ accordingly. From Observation: $\mathrm{conv}(U) \cap \mathrm{conv}(V) \neq \emptyset$.

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Solve linear feasibility program

$$\sum_{u \in U} \lambda_u u = \sum_{v \in V} \kappa_u v$$
$$\sum_{u \in U} \lambda_u = 1, \sum_{v \in V} \kappa_u = 1$$
$$\lambda \geq 0, \kappa \geq 0,$$

to obtain

$$x := \sum_{u \in U} \lambda_u u = \sum_{v \in V} \kappa_u v \in P \cap Q.$$

An exact algorithm

Alternating Linear Minimizations for Polytopes

Algorithm Alternating Linear Minimizations (ALM) [exact version]

Input: Points $x_0 \in P$, $y_0 \in Q$, LMO over $P, Q \subseteq \mathbb{R}^n$
Output: Iterates $x_1, y_1 \ldots \in \mathbb{R}^n$

```
for t = 0 to ... do
    u_t ← argmin_{x∈P} ⟨x_t − y_t, x⟩
    x_{t+1} ← x_t + 2/(t+2) · (u_t − x_t)
    v_t ← argmin_{y∈Q} ⟨y_t − x_{t+1}, y⟩
    y_{t+1} ← y_t + 2/(t+2) · (v_t − y_t)
    if t = 2^k for some k then
        if min_{x∈P,y∈Q} ⟨x_{t+1} − y_{t+1}, x − y⟩ > 0 then
            return "disjoint" and certificate ⟨x_{t+1} − y_{t+1}, x − y⟩ > 0
        else
            Solve linear feasibility program.
            if feasible then
                return a solution x ∈ P ∩ Q
```

An exact algorithm: Guarantees

Alternating Linear Minimizations for Polytopes

Basically we pay a factor of 2 in iterations for making exact.

Proposition (Exact variant)

Let P, Q be polytopes with diameters D_p and D_Q, respectively. Executing exact ALM variant:

1. *If $P \cap Q \neq \emptyset$, then after no more than*

$$\frac{16(1+2\sqrt{2})(D_P^2+D_Q^2)}{\varepsilon_{PQ}^2}$$

   block-LMO calls, the algorithm returns $x \in P \cap Q$.

2. *If $P \cap Q = \emptyset$, then after no more than*

$$16(1+2\sqrt{2})(D_P^2+D_Q^2)\frac{(D_P+D_Q)^2}{\operatorname{dist}(P,Q)^4}$$

   block-LMO calls the algorithm certifies $P \cap Q = \emptyset$.

Note. We counted the resolution of one feasibility LP as one block-LMO.

Thank you!

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