Fast Algorithms for Packing Proportional Fairness and its Dual

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Proportional fairness

Suppose we have a resource allocation problem where a single resource has to be allocated to n agents. One could try and maximize the amount of the resource allocated, but this could strongly favor some agents in the detriment of others.

A resource allocation satisfies **proportional fairness** (also called 1-**fairness**) if it maximizes the sum of the log-utilities of each agent: $f(x) = \sum_{i=1}^{n} \log x_i$.

Proportional fairness is the only utility function satisfying several natural *fairness axioms* (Bertsimas et al. 2011). We are interested in studying this fairness criterion for the case of positive polyhedra (packing problems), which appear naturally in various resource allocation problems like network flows. This is a set of the form $\mathscr{P} = \{x \in \mathbb{R}^n_{\geq 0} : Ax \leq 1_m, A \in \mathbb{R}^{m\times n}_{\geq 0}\}$.

Definition Packing proportional fairness and its dual

Let $A \in \mathbb{R}^{m\times n}_{\geq 0}$ be a nonnegative matrix. We study the following two problems:

$$\textbf{1-fair packing}: \quad \max_{x\in\mathbb{R}^n_{\geq 0}} \{f(x) \stackrel{def}{=} \sum_{i=1}^{n} \log x_i : Ax \leq 1_m\}. \qquad (1)$$

$$\textbf{Dual 1-fair packing}: \quad \min_{\lambda\in\Delta^n} \{g(\lambda) \stackrel{def}{=} -\sum_{i=1}^{n} \log(A^T\lambda)_i - n\log n\}. \qquad (2)$$

Paper	Problem	Iterations
Beck, Nedic, Ozdaglar, Teboulle, (2014)	Primal	$O(\rho^2 mn/\varepsilon)$
Marašević, Stein, Zussman (2015)	Primal	$\widetilde{O}(n^5/\varepsilon^5)$
Diakonikolas, Fazel, Orecchia (2020)	Primal	$\widetilde{O}(n^2/\varepsilon^2)$
This work	Primal	$\widetilde{O}(n/\varepsilon)$
Beck, Nedic, Ozdaglar, Teboulle, (2014)	Dual	$\widetilde{O}(\rho\sqrt{mn/\varepsilon})$
This work	Dual	$\widetilde{O}(n^2/\varepsilon)$

Primal problem: Regularization and coupling

We optimize a regularized function for the 0-fair packing problem similar to the one in (Diakonikolas et al. 2020). First by an exp reparametrization transforming the constraints into a barrier function:

$$f_r(x) \stackrel{def}{=} -\sum_{i\in[n]} x_i + \frac{\beta}{1+\beta}\sum_{i\in[m]} (A\exp(x))_i^{\frac{1+\beta}{\beta}}.$$

Here β depends on n, m, ε. Our algorithm is deterministic & distributed.

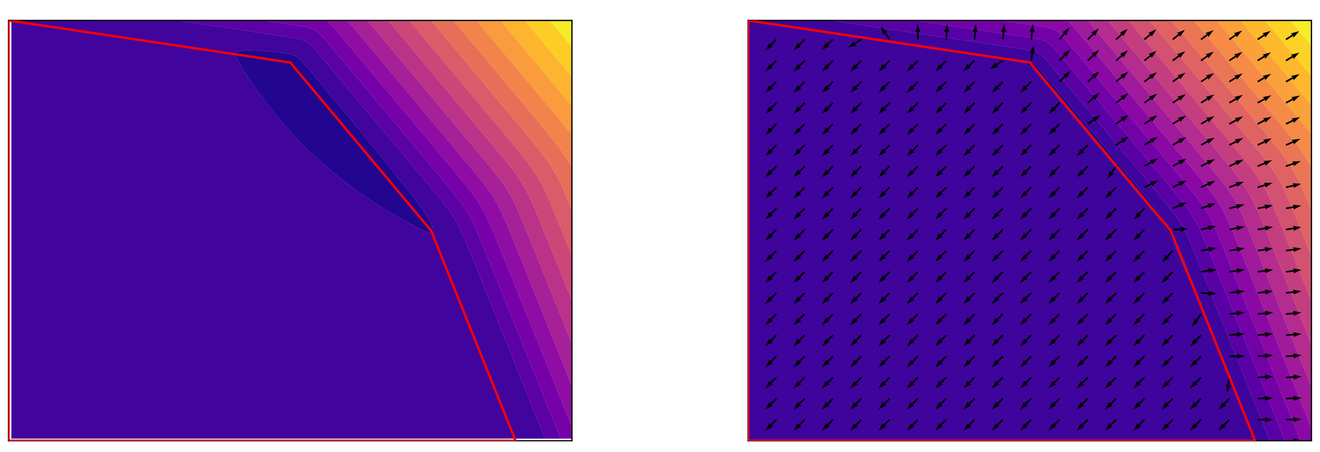

Figure 1: The regularized objective f_r uses a non-standard barrier function. Here, on the original space, we plot $\log(f_r)$ (left) and $\log(\|\nabla f_r\|)$ (right) with normalized gradient arrows (right).

Theorem 1

Let $\varepsilon \leq n/2$ and let $\bar{x}^*$ be the optimum solution of Problem (1). Our algorithm computes a point $y^{(T)} \in B$ such that $f_r(y^{(T)}) - f_r(x_r^*) \leq \varepsilon$ in a number of iterations $T = \widetilde{O}(n/\varepsilon)$. Besides, $\hat{x} \stackrel{def}{=} \exp(y^{(T)})/(1+\varepsilon/n)$ is a feasible point of Problem (1), i.e., $A\hat{x} \leq 1_m$, and $f(\bar{x}^*) - f(x) \leq 5\varepsilon = O(\varepsilon)$.

The algorithm uses a non-standard **linear coupling** (Allen-Zhu et al., 2014) via gradient truncation. For Δ s.t. $\Delta_j = \eta_j \min\{\nabla_j f_j(x), 1\}$ and small η_j, we have $\langle \nabla f_r(x), \Delta\rangle \geq f_r(x) - f_r(x-\Delta) \geq \frac{1}{2}\langle \nabla f_r(x), \Delta\rangle \geq 0$ and this strong descent condition is enough to compensate for the regret of a mirror descent run on truncated gradients in a box B, plus the regret coming from only using the truncation.

Dual problem: The centroid map and PST

Intuitively, Problem (2) is about finding the simplex minimizing volume with a fixed corner in the positive orthant that covers $\mathscr{P}$ (check our paper for an application to linear programming). We identify the covering constraints $\langle h, x\rangle \leq 1$ with the (dual) point $h \in \mathbb{R}^n_{\geq 0}$.

Definition

- $\mathscr{D}^+$ is the set of constraints (i.e., dual points) feasible in all $\mathscr{P}$.
- **Centroid map** $c: \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$, $c(h) = \left(\frac{1}{nh_1}, \dots, \frac{1}{nh_n}\right)$.

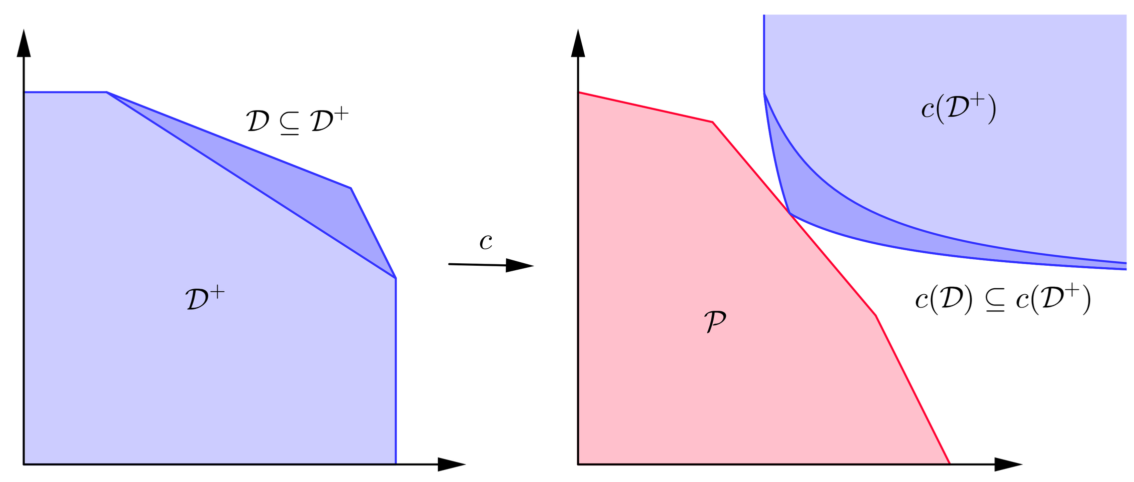

Figure 2: Left: c maps dual points (i.e., feasible constraints in $\mathscr{P}$) to primal points. Right: $\mathscr{P}$ with the image of $\mathscr{D}^+$ under c.

The solution is the unique point in the intersection $\mathscr{P} \cap c(\mathscr{D}^+)$. This motivates the study of the following **proxy problem**:

$$\min_{p\in c(\mathscr{D}^+)} \{\hat{g}(p) \stackrel{def}{=} \max_{i\in[m]} \langle A_i, p\rangle\}. \qquad (3)$$

We solve (3) as a *linear feasibility problem* on $c(\mathscr{D}^+)$ via a variant of the Plotkin-Shmoys-Tardos (PST) algorithm with a novel geometric oracle.

Theorem 2

Let $\varepsilon \in (0, n(n-1)]$. Our algorithm finds a linear combination of the rows of A, $\lambda \in \Delta^m$ such that $\hat{g}(c(\lambda^T A)) \leq 1 + \varepsilon/n$ (i.e., an (ε/n)-approximate solution of Problem 3) after $\widetilde{O}(n^2/\varepsilon)$ iterations. Furthermore, this same solution is an ε-minimizer solution of Problem 2.

The geometric oracle

Given a covering constraint $h = A^T\lambda$, for weights $\lambda \in \Delta^m$ that change with a MWs algorithm, and given an oracle that finds $x \in c(\mathscr{D}^+)$ s.t. $\langle h, x\rangle \leq 1$, if we can guarantee the losses for MWs $\langle A_i, x\rangle - 1$ are in $[-\tau, \sigma]$, then PST ensures convergence in $O(\sigma\tau/\varepsilon^2)$.

We obtain an improved algorithm by implementing an oracle using δ-minimizer of $\hat{g}$, that has τ_δ, σ_δ that decrease with δ. We can find a $\delta/2$-minimizer in $O(\tau_\delta\sigma_\delta/(\delta/2)^2)$, and repeat until $\delta < \varepsilon/n$. Total complexity is $O(n^2/\varepsilon)$.

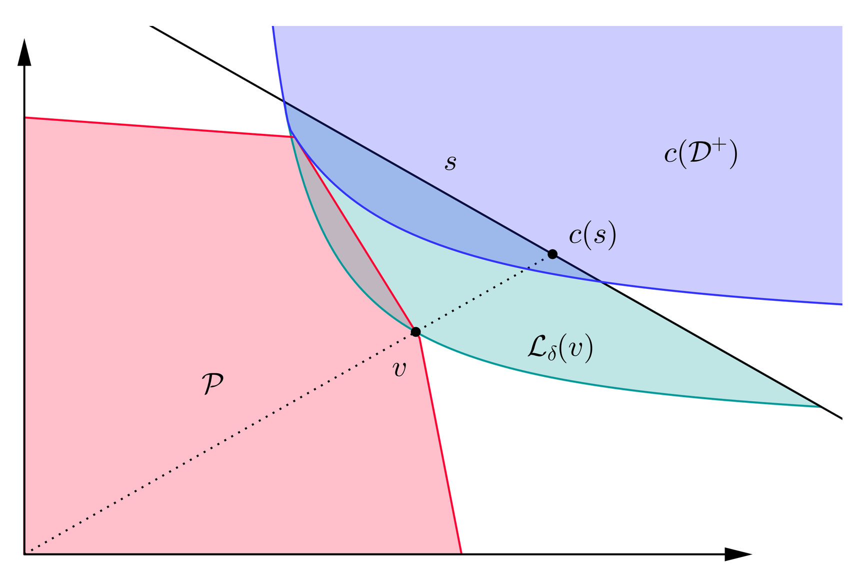

Figure 3: If we have a δ-minimizer $c(s)$, the optimum is in *the lens* $L_\delta(v)$, which is smaller the lower δ is. The oracle uses $c(\cdot)$ of a cvx. combination of s and the query h.